UNIVERSITY of the WESTERN CAPE

The foundations of locale theory

Author: Joshua David Passmore
Supervisor: Professor David Holgate

July 2025

A thesis submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematical Science within the Department of Mathematics and Applied Mathematics at the University of the Western Cape

Abstract

Locale theory (or point-free topology) may be regarded as the study of topology on a lattice-theoretic foundation. Instead of topological spaces, the objects of study are called frames (or locales); these are complete lattices which satisfy a particular infinite distributive law. Frames and their homomorphisms offer one perspective of point-free topology, another perspective is given by locales and their morphisms (localic maps). The theory of locales generalises sober space topology; moreover, there is a categorical duality between the so-called spatial frames and sober spaces. Locale theory genuinely offers new insights and results in topology, for instance, sublocales (i.e. generalised subspaces) are better behaved than their classical counterparts: every locale contains a least dense sublocale – a result which has no analogue in point-set topology – this has the consequence that there are more point-free spaces than classical spaces, i.e. there exist locales (the non-spatial ones) which do not arise from topological spaces. Since frames are algebras, frames may be presented by generators and relations; in our case the generators form a meet-semilattice and the relations are encoded in a coverage on the generators – a meet-semilattice equipped with a coverage is called a site, and every site canonically freely generates a frame (for example, the frame of ideals of a distributive lattice is just a special case of a freely generated frame over a site). Compactness of locales is a point-free invariant, and there is a localic Kuratowski-Mrówka Theorem which characterises compact locales via closed projections from a binary coproduct of frames – this offers a categorical (or extrinsic) characterisation of localic compactness.

Acknowledgements

I am grateful to my supervisor, David Holgate, for all the support and encouragement he has given me along the way of my mathematical journey; he has been incredibly kind and understanding towards me. Prof Holgate is the research chair of *Topology for Tomorrow* (T4T), an ongoing collaborative mathematics project at the University of the Western Cape, which funded my studies during the duration of my Master's – I wish to thank David and T4T for this financial support. I have made many friends through T4T and UWC, they have made my university experience positively memorable. I must thank Ando Razafindrakoto for his help with categorical concepts, particularly adjoint functors (these play an important rôle in the thesis), my discussions with him were very fruitful. Many point-free insights were made through conversations with my good friend Ana Belén – Ana has been a great friend to me, and she greatly helped me with my learning of point-free topology. Life is not all about mathematics, and I must thank those closest to me for keeping my life balanced, particularly my dear friend Inken. Lastly, I thank my family for being my foundation in life.

Declaration

I declare that *The foundations of locale theory* is my own work, that it has not been submitted before for any degree or assessment in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by means of complete references.

Joshua David Passmore

Date: July 26, 2025

Contents

Chapter 0

Introduction

Most university students who pursue an undergraduate degree in pure mathematics will encounter the subjects of analysis (both real and complex), metric spaces, and possibly elementary point-set topology. It is certainly the case that a graduate programme in pure mathematics will include a course on topology; we learn that a topological space is a set of points equipped with a family of open sets (a topology), and our spatial intuition becomes ingrained in this way of thinking (i.e. working with points and open sets). One perspective that the lecturer may portray to the students is that the notion of topological space generalises that of metric space, where the distance function is forgotten and there is an emergence of spaces with no concept of distance at all – it is then the open sets (which live on the points) that play a foundational rôle. However, after the development of lattice theory in the 1930s and 1940s, there came a significant generalisation of the notion of topological space, this is the notion of locale (this can be taken a step further, in its geometric aspects the notion of topos generalises that of locale – for more on this see [16] – though we won't be exploring toposes here). With locales we forget about the points in spaces: the lattices of open sets become the primitive notion (see [14]). The topological nature of a locale lies in its 'lattice of opens', and the points (if any) live on the open sets rather than the other way round (cf. [15]). Point-free topology (also known as locale theory) is the study of topology on a lattice-theoretic foundation, where the objects of study are colloquially referred to as 'spaces without points' (i.e. locales). However, point-free topology has two flavours to it: one algebraic, and the other topological. From the algebraic perspective, the objects of study are called frames and the structure-preserving maps are called frame homomorphisms – we thus have a concrete category called the category of frames. The topological features of point-free topology display themselves when one works with the opposite of the category of frames – this is known as the category of locales. Thus frames and locales are extensionally the same thing, the only difference is in the morphisms: frames arise algebraically and locales topologically.

Before we continue, let us point out the following: the reader is assumed to understand the basic theory of posets and lattices, as well as some foundational category theory, in particular, one should be comfortable working with categories, functors, natural transformations, and (co-)products. We will cover adjoint functors in the proceeding chapter. Some knowledge of elementary point-set topology would be useful too (see [22] for the required background). For background in lattice theory, we refer the reader to the first two chapters of [6] (the fourth chapter is also useful for insight into distributive lattices and Boolean algebras), and our main reference for category theory is Leinster's excellent textbook [21]. If the reader desires a terse 'crash course' in lattices and categories, then we recommend the first chapter of [13]. In this introductory chapter of the thesis we will outline some of the historical developments of point-free topology; our references for this are given by

[14, 15, 16].

Modern pure mathematics may broadly be partitioned into two (interconnected) parts: algebra (the study of operations on structured sets) and geometry[1] (the study of spatial relations – see [31, Introduction]). Indeed, categorically speaking, they are dual to each other. As a simple concrete example of such a duality, we can consider the category **Set** (of sets and their functions) and the category **CABool** (of complete atomic Boolean algebras and their homomorphisms). Here **Set** is viewed as geometric and **CABool** is viewed as algebraic. Then it is straightforward to see that **CABool** is dual to **Set** (i.e. $\mathbf{Set}^{\mathrm{op}} \simeq \mathbf{CABool}$, where '$\simeq$' denotes equivalence of categories): the powerset functor $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{CABool}^{\mathrm{op}}$ (which sends each set A to its powerset $\mathcal{P}(A)$, and each set function $f : A \longrightarrow B$ to its induced preimage function $f^{-1}[-] : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$) is an equivalence (it is fully faithful, and essentially surjective on objects) – for further details on this duality the reader should consult [40]. What is striking about this fact is that **Set** is quite an ordinary category (sets and their functions aren't very structured), while, on the other hand, complete atomic Boolean algebras and their homomorphisms are *very* structured. It is the simple act of turning the arrows round which produces a dramatically different category – for the most part, this is generally true about categories. For instance, in [15] Johnstone states:

"*most categories of practical importance in mathematics (with two important classes of exceptions: abelian categories and categories of relations) are very different from their opposites, and our categorical intuition (largely conditioned by our experience of the categories of sets and of topological spaces) is strongly non-self-dual, so that the simple act of turning the arrows round produces a new insight into the nature of the objects and the way in which they interact*."

The duality between **Set** and **CABool** is just one of the many examples throughout pure mathematics which amplify the slogan 'algebra is dual to geometry' (we will meet more examples later on). If we think historically for a moment, the need for an algebraic approach to geometry is significant, for example; vector spaces emerged as an abstraction of the Euclidean space that was used in classical physics. The field of linear algebra, which finds useful applications in quantum mechanics (the state space of a quantum mechanical system is typically an infinite dimensional Hilbert space – see [8]), developed with this abstraction of Euclidean space and is perhaps one of the first examples of the connection between algebra and geometry. A common theme that we will explore further is the intertwined relationship between algebra and geometry.

0.1 The origins of point-free topology

There is already a detailed article on the history of point-free topology written by Johnstone (see [16]), our goal here is to provide a summary of that article and highlight the emergence of locale theory as an autonomous subject in its own right.

The roots of point-set topology can be traced back to the development of real analysis in the nineteenth century. However, a great leap forward came in 1906 with Maurice Fréchet's doctoral dissertation in which he introduced metric spaces that enabled the abstraction of analysis on other 'exotic' spaces as opposed to just the reals or complex numbers (this was particularly the case in the 1920s with the development of normed vector spaces in which the norm induces the metric). The breakthrough with metric spaces is that we can begin to talk

[1] We include the subject of analysis as a subset of geometry.

abstractly about distances between points in a space, and hence write down precise definitions for concepts such as convergence and continuity in terms of the distance function. Fréchet did not introduce the notion of open set in metric spaces, though he did work with closed sets: he defined a set to be closed iff it contained all its limit points (i.e. iff it contained the limit of every convergent sequence inside it). Felix Hausdorff extended the work of Fréchet: in 1914 Hausdorff released his book *Grundzüge der Mengenlehre* (translated as 'Basics of Set Theory') [9, 10], in it he introduced as a primitive notion the concept of a neighbourhood of a point and thus freed the restrictive notion of distance. A space then consisted of a set of points and a set of neighbourhoods. The use of neighbourhoods enabled Hausdorff to define the concept of open set in an abstract sense (nowadays we may declare a set to be open iff it is a neighbourhood of each of its points), and he was indeed able to show that any union of open sets is open and any finite intersection of open sets is open. Hausdorff was thus the first mathematician to place emphasis on the importance of the notion of open set in the study of continuity properties in abstract spaces [14]. Incidentally, Hausdorff had defined his spaces with some degree of separation between points: any two distinct points could be separated through disjoint neighbourhoods (this is of course what is now referred to as the Hausdorff separation axiom). For more on the early history of point-set topology and analysis, see [23].

Shortly after Hausdorff's time, it became well-known that a topological space is something which is defined by a set of points as well as a lattice of open sets. Subsequently, many 'topological properties' (e.g. compactness and connectedness) that were familiar in analysis were extended to topological spaces, and theorems regarding these properties were constructed. There are in particular two theorems that stand out to us:

Theorem 0.1.1. *A topological space X is compact iff, for any space Y, the projection $\pi_Y : X \times Y \longrightarrow Y$ is closed.*

Theorem 0.1.2. *Let X be a topological space. If X is Hausdorff, then compact subspaces of X are closed.*

Most textbooks on point-set topology (e.g. [4, 22]) will provide a proof for Theorem 0.1.2, while Theorem 0.1.1 (known as the Kuratowski-Mrówka Theorem for compact spaces) was first proved in [24]. Both of these theorems can be extended to the context of point-free topology – see for instance [35, 42] – and it is one of the goals of this thesis to construct the point-free proof of Theorem 0.1.1.

Until the 1950s, topological spaces were understood through both their points and their open sets: the point-free perspective (i.e. the philosophy that a space *is* a lattice of open sets) had not yet emerged. The field of lattice theory (as pioneered by Garrett Birkhoff in his book *Lattice Theory* [3]) only saw development in the 1930s and 1940s. A pivotal moment came with Marshall Stone's representation theorem for Boolean algebras in the years of 1936 and 1937 (see [37, 38]) – which may be regarded as the initial impetus for the lattice-theoretic approach to topology – we now refer to this result as *Stone duality*; in today's language this says that the category of Boolean algebras is dual to the category of totally disconnected compact Hausdorff spaces (see [21]), this is consistent with the idea that 'algebra is dual to geometry'. What is remarkable about Stone's work is that he proved his result before the birth of category theory in the 1940s. Stone duality tells us that topology cannot be separated from algebra.

Around the year 1957, Ehresmann and Bénabou observed that, in some sense, the open sets in spaces are more fundamental than the points [14]. They began studying certain complete lattices which satisfied an infinite distributive law – they called such lattices *local lattices* (today we now refer to them as *frames* or *locales*). It was noted that every topological space gives rise to a frame, but not all frames come from topological spaces; thus frames were seen to 'generalise' topological spaces. The name 'frame' for describing these local lattices made its first appearance in the paper [7] (in 1966) by Dowker and Papert-Strauss. In [7], the authors define the category **Frm** as the category whose objects are frames and whose morphisms are *frame homomorphisms* (i.e. those functions which preserve all joins and finite meets). It was observed that there is a canonical *contravariant* functor **Top** $\longrightarrow$ **Frm** (where **Top** is the category of topological spaces and continuous maps); this contravariance lead to Isbell (in the year 1972, see [11]) to define the category **Loc** of *locales* as the formal opposite of **Frm**, thus making **Loc** a covariant 'extension' of **Top**. In [11], Isbell observed that **Loc** is in fact an improvement of **Top**; this improvement essentially all comes down to the fact that in **Loc** every locale contains a least dense sublocale (a fact which has no analogue in **Top**). Locale theory (or point-free topology) may be understood as the study of **Frm** and **Loc**; however, **Loc** is formally not a concrete category – the morphisms are 'reversed' frame homomorphisms – in [31] localic maps are introduced as the right adjoints of frame homomorphisms, and **Loc** is there thought of as the concrete category with locales as objects and localic maps as morphisms.

0.2 Synopsis of the thesis

The goal of this thesis is to provide a general survey of locale theory (i.e. the study of topology on a lattice-theoretic foundation), starting from spaces and then moving to frames. The final chapter sees applications of the developed theory towards point-free compactness, closed localic maps, and the localic Kuratowski-Mrówka Theorem.

Chapter 1

The mathematical preliminaries are covered in Chapter 1; this consists of Galois connections, Heyting algebras, and adjoint functors; those who are already comfortable with this material can skip this chapter. Here, we first start with Galois connections (and some immediate results which follow from the definition) and then cover the important 'adjoint functor theorem for complete lattices'. Heyting algebras follow immediately from Galois connections, and the Heyting implication provides a useful example of a right adjoint. We note that a complete lattice is a Heyting algebra iff a certain infinite distributive law holds (the frame distributivity). We move onto the discussion of the theory of adjoint functors in some detail – an adjuntion is introduced as a natural bijection between particular hom-sets. We offer alternative (but equivalent) characterisations of adjoint functors in terms of the unit and counit natural transformations, as well as universal properties. We emphasise that every adjunction restricts to an equivalence between full subcategories in a canonical way, here the notion of idempotent adjunction plays a special rôle.

Chapter 2

Our exposition of the theory of locales begins in Chapter 2; we first demonstrate how one abstracts the definition of a frame from the lattice of open sets of a topological space, and, in a similar manner, how one obtains a frame homomorphism from a continuous map (frames and their homomorphisms yield the category **Frm**). The category of locales is defined as **Loc** := **Frm**$^{\mathrm{op}}$, this allows us to define a canonical functor $\mathcal{O}$: **Top** $\longrightarrow$ **Loc**. Locales are viewed as a kind of space, and thus the notion of point of a locale (frame) is defined – we offer alternative descriptions of points. The 'point-set' of a locale yields a topological space in a canonical way;

this defines a functor pt : **Loc** $\longrightarrow$ **Top** which we demonstrate as being right adjoint to O. The adjunction $O \dashv$ pt restricts to an equivalence between the so-called 'sober spaces' and 'spatial locales'. The spatial locales are precisely those locales which arise as the open-set lattices of topological spaces, thus non-spatial locales are inherently of particular interest: we show that a Boolean frame is spatial iff it is atomic (as a complete Boolean algebra) – this provides an ample supply of non-spatial frames. We end the chapter by providing various characterisations for sobriety, and show that every Hausdorff space is sober, and every sober space is T_0. It turns out that sobriety is incomparable with the T_1 separation axiom.

Chapter 3

The subobjects in **Loc** may initially be described as certain closure operators called nuclei (a nucleus on a locale A is a closure operator $n : A \longrightarrow A$ which preserves finite meets). A sublocale of a locale A is then defined as the fixset of a nucleus $n : A \longrightarrow A$; the content of Chaper 3 is devoted to the study of sublocales. We show that a sublocale can be characterised as a subset closed under both arbitrary meets and the Heyting implication in the ambient locale. A useful result is that any localic map naturally factors through a sublocale as an epimorphism followed by an extremal monomorphism. In order to motivate the definitions of open and closed sublocale, we demonstrate how any subspace of a topological space induces a sublocale of the corresponding open-set frame. The fundamental result of Chapter 3 is Theorem 3.9.3 which states that every locale contains a least dense sublocale – this result is one of the many improvements of topology as studied in the category **Loc**. We end the chapter with a representation theorem for frames which states that every frame is isomorphic to a subframe of a complete Boolean algebra.

Chapter 4

The algebraic aspects of frames are explored in Chapter 4; the fact that frames are equationally presentable (with the existence of free frames over sets) has a number of pleasant consequences (see Theorem 3.1.6). We are mostly interested in the construction of free frames over meet-semilattices: the free frame over a meet-semilattice A is the frame $\mathfrak{D}(A)$ of down-sets of A. We show that there is a canonical way to 'freely generate' a frame over a meet-semilattice; with the notions of coverages and sites, the concept of C-ideal is defined – the set C-Idl(A) of all C-ideals of a meet-semilattice A is a sublocale of $\mathfrak{D}(A)$ and is 'freely generated'. Our proof of Propostion 4.2.12 seems to be a novel way of proving that C-Idl(A) is a sublocale of $\mathfrak{D}(A)$ – the proof makes use of our Lemma 4.2.11.

Chapter 5

The final chapter is where we see the applications of locale theory in the study of compactness. The goal of Chapter 5 is to prove the localic version of the celebrated Kuratowski-Mrówka Theorem (see [24]); our final result (Theorem 5.4.10) is not a new fact, but our approach to arriving there may be more straightforward than that of [20, 35]. In particular, we discovered a novel method for proving Theorem 5.2.4 (this was previously proved in [20], but the Axiom of Choice was used there) – our method of proof is in fact constructive (i.e. we do not use Choice nor excluded middle). Closed localic maps are discussed in detail, in particular closed localic projections. Using the notion of the *unit decomposition property* (briefly, UD) – as introduced in [35] – we demonstrate that a locale is compact iff it satisfies UD. Finally, we construct an extrinsic characterisation for localic compactness: a frame A is compact iff, for all frames B, the localic projection $\pi_B : A \oplus B \longrightarrow B$ is closed – this is the localic (or point-free) Kuratowski-Mrówka Theorem.

Chapter 1

Preliminaries

In order to *do* locale theory, one needs a prior understanding of posets and lattices, as well as category theory. The aim of this chapter is to introduce the mathematical machinery (from lattice theory and category theory) that we will be using throughout the rest of the thesis.

Galois connections and Heyting algebras are of the utmost importance in locale theory: we will later see that we can use Galois connections to *define* localic maps from frame homomorphisms (frame homomorphisms are left adjoints and localic maps are right adjoints), moreover the Heyting implication appears in the definition of sublocale and becomes useful when navigating sublocales since the Heyting implication is a right adjoint.

Adjoint functors are ubiquitous in mathematics, and indeed in locale theory: the interaction between topological spaces and locales results in an idempotent adjunction; moreover, there are multiple free-forgetful adjunctions between certain categories of semilattices and frames. We will meet examples of adjoint functors along the way of this investigation, and thus we introduce them in this chapter.

1.1 Galois connections and Heyting algebras

A Galois connection between posets is a special case of something more general, that is, an adjunction between categories. Before looking at the general theory of adjoint functors, we will restrict ourselves to adjoint monotone maps. This will provide us with a firm grounding in the mathematics of adjoints: much of the phenomena that is present in adjunctions between categories can already be seen in Galois connections between posets.

1.1.1 Galois connections

We could begin the definition of Galois connection with respect to preordered sets, however, we will essentially only be working with posets (and lattices, complete lattices, etc.) hence we will provide the definition of Galois connection in terms of posets. Our main reference for this section is [31].

Definition 1.1.1. Let X and Y be posets, and consider functions $X \underset{g}{\overset{f}{\rightleftarrows}} Y$. We say that f and g are in a *Galois connection* or are *Galois adjoint* (with f *left adjoint to* g, g *right adjoint to* f, and write

$f \dashv g$) iff

$$(\forall x \in X)(\forall y \in Y) \qquad f(x) \leqslant y \iff x \leqslant g(y).$$

As for notation, we will often denote a Galois connection by $X \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Y$ to mean that the function f is left adjoint to the function g.

Observation 1.1.2. Consider a Galois connection $X \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Y$. We can view f and g as functions between the opposite posets: $X^{\mathrm{op}} \underset{g}{\overset{f}{\underset{\longleftarrow}{\longrightarrow}}} Y^{\mathrm{op}}$. Let $x \in X$ and $y \in Y$, then we have

$$\begin{aligned} f(x) \leqslant y &\iff x \leqslant g(y) \\ \text{iff} \quad y \leqslant^{\mathrm{op}} f(x) &\iff g(y) \leqslant^{\mathrm{op}} x. \end{aligned}$$

We can conclude that we have a Galois connection $X^{\mathrm{op}} \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} Y^{\mathrm{op}}$, i.e. f is *right* adjoint to g in this dual case. It follows that the notions of left and right adjoint are dual to one another: this is important, as we will be able to deploy the principle of duality in our arguments about Galois adjoints.

Proposition 1.1.3. *Let* $X \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Y$ *be a Galois connection. Then f and g are both monotone.*

Proof. Take $x_1 \leqslant x_2$ in X. We have

$$\begin{aligned} & f(x_2) \leqslant f(x_2) \\ \iff & x_2 \leqslant g(f(x_2)) \quad \text{(because } f \dashv g\text{)} \\ \implies & x_1 \leqslant g(f(x_2)) \\ \iff & f(x_1) \leqslant f(x_2) \quad \text{(because } f \dashv g\text{).} \end{aligned}$$

Thus f is monotone. That g is monotone follows dually. ■

So for $X \underset{g}{\overset{f}{\underset{\longleftarrow}{\longrightarrow}}} Y$ to be a Galois connection between posets, it is necessary that f and g both be monotone. From now on, whenever we are dealing with Galois connections, we will always make it explicit that we are working with monotone maps.

Proposition 1.1.4. *Let $X \xrightarrow{f} Y$ be a monotone map between posets. Suppose that a right adjoint $Y \xrightarrow{g} X$ of f exists. Then g is unique.*

Proof. Let $Y \xrightarrow{g'} X$ also be a right adjoint of f. For all $x \in X$ and $y \in Y$ we have $x \leqslant g(y)$ iff $f(x) \leqslant y$ iff $x \leqslant g'(y)$. Now $g(y) \leqslant g(y)$, so $g(y) \leqslant g'(y)$. Also, $g'(y) \leqslant g'(y)$, so $g'(y) \leqslant g(y)$. By antisymmetry we conclude that $g(y) = g'(y)$. It follows that g is the unique right adjoint of f. ■

By duality we obtain

Fact 1.1.5. *Let $Y \xrightarrow{g} X$ be a monotone map between posets. Suppose that a left adjoint $X \xrightarrow{f} Y$ of g exists. Then f is unique.*

Proposition 1.1.6. *Let* $\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$ *be monotone maps between posets. Then $f \dashv g$ iff $f \cdot g \leqslant \mathrm{id}_Y$ and $\mathrm{id}_X \leqslant g \cdot f$.*

Proof. ($\Rightarrow$) Suppose $f \dashv g$. For all $y \in Y$, we have $g(y) \leqslant g(y)$, and hence $f(g(y)) \leqslant y$. Dually, for all $x \in X$, $x \leqslant g(f(x))$.

($\Leftarrow$) Suppose that $f \cdot g \leqslant \mathrm{id}_Y$ and $\mathrm{id}_X \leqslant g \cdot f$. Take $x \in X$ and $y \in Y$.

- Suppose that $f(x) \leqslant y$. Then $g(f(x)) \leqslant g(y)$, since g is monotone. But $x \leqslant g(f(x))$, so $x \leqslant g(y)$.
- Suppose that $x \leqslant g(y)$. Then $f(x) \leqslant f(g(y))$, since f is monotone. But $f(g(y)) \leqslant y$, so $f(x) \leqslant y$.

Thus $f(x) \leqslant y$ iff $x \leqslant g(y)$. ■

Proposition 1.1.6 shows us that Galois connection is weaker than order isomorphism: every order isomorphism yields a Galois connection, but not every Galois connection yields an order isomorphism. As an example, consider a function between *sets* $f : X \longrightarrow Y$. This induces a so-called *image-preimage adjunction* between the powerset lattices:

$$f[-] : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^{-1}[-] \tag{1.1}$$

For each $A \subseteq X$ and $B \subseteq Y$, we have $f[A] \subseteq B$ iff $A \subseteq f^{-1}[B]$, and thus $f[-] \dashv f^{-1}[-]$. But, in general, we cannot expect (1.1) to be an isomorphism.

Remark 1.1.7. $f^{-1}[-]$ also has a right adjoint, namely $Y \setminus f[X \setminus -]$.

As a consequence of Proposition 1.1.6, we obtain

Corollary 1.1.8. *Let* $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$ *be a Galois connection between posets. Then $fgf = f$ and $gfg = g$.*

Proof. Take $x \in X$. Then $x \leqslant g(f(x))$, so $f(x) \leqslant f(g(f(x)))$. But we also have $f(g(f(x))) \leqslant f(x)$. Hence $f(g(f(x))) = f(x)$. That $gfg = g$ follows by duality. ■

Corollary 1.1.9. *Let* $\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$ *be monotone maps between posets that are in a Galois connection (i.e. $f \dashv g$ or $g \dashv f$). Then f is injective iff g is surjective.*

Proof. ($\Rightarrow$) Let f be injective. Take $x \in X$. Then $f(g(f(x))) = f(x)$, so $g(f(x)) = x$. Thus g is surjective.

($\Leftarrow$) Let g be surjective. Let $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. There exist $y_1, y_2 \in Y$ such that $x_1 = g(y_1)$ and $x_2 = g(y_2)$. So $f(g(y_1)) = f(g(y_2))$. Then $x_1 = g(y_1) = g(f(g(y_1))) = g(f(g(y_2))) = g(y_2) = x_2$. Thus f is injective. ■

Remark 1.1.10. We note that, as a consequence of Corollary 1.1.9, for a Galois connection $\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$ (i.e. $f \dashv g$ or $g \dashv f$) if f is injective (equivalently, if g is surjective) then $gf = \mathrm{id}_X$.

Observation 1.1.11. Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ (with $f \dashv g$) be a Galois connection between posets. This induces a Galois connection between suitable subposets,

$$\begin{tikzcd} g[Y] \arrow[r, bend left, "\overline{f}"] \arrow[r, phantom, "\perp"] & f[X], \arrow[l, bend left, "\overline{g}"] \end{tikzcd}$$

where $\overline{f} : g[Y] \ni g(y) \longmapsto f(g(y))$ and $\overline{g} : f[X] \ni f(x) \longmapsto g(f(x))$. Straightforward verification reveals that $\overline{f}$ and $\overline{g}$ are both bijective; moreover, $\overline{f} \cdot \overline{g} = \mathrm{id}_{f[X]}$ and $\overline{g} \cdot \overline{f} = \mathrm{id}_{g[Y]}$ (i.e. $(\overline{f})^{-1} = \overline{g}$). We thus have $g[Y] \cong f[X]$, and conclude that every Galois connection between posets induces an isomorphism of suitable subposets.

Proposition 1.1.12. *Let* $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$ *be a Galois connection between posets. Then f preserves all existing joins, and g preserves all existing meets.*

Proof. Let $A \subseteq X$, and suppose that $\bigvee A$ exists in X. Put $s = \bigvee A$. Since f is monotone, $f(s)$ is an upper bound of $f[A]$ in Y. Let $y \in Y$ be an upper bound of $f[A]$. For each $a \in A$, $f(a) \leqslant y$, and hence $a \leqslant g(y)$. Thus $g(y)$ is an upper bound of A. But s is the least upper bound of A, so $s \leqslant g(y)$, and it follows that $f(s) \leqslant y$. This shows that $f(s)$ is the least upper bound of $f[A]$ in Y, i.e. $f(s) = f(\bigvee A) = \bigvee f[A]$. That g preserves all existing meets follows by duality. ■

The most important result in this section is the so-called 'adjoint functor theorem for complete lattices'. This is presented in the following two theorems.

Theorem 1.1.13. *Let X and Y be complete lattices, and let $X \xrightarrow{f} Y$ be a monotone map. Then f is a left adjoint iff f preserves all joins.*

Proof. ($\Rightarrow$) This is Proposition 1.1.12.

($\Leftarrow$) Let f preserve all joins. Define a function $f_* : Y \longrightarrow X$ by $f_*(y) = \bigvee\{x \in X \mid f(x) \leqslant y\}$. Take $a \in X$ and $y \in Y$.

- Suppose $f(a) \leqslant y$. Then $a \in \{x \in X \mid f(x) \leqslant y\}$, so $a \leqslant f_*(y)$.
- Suppose $a \leqslant f_*(y)$. Put $A = \{x \in X \mid f(x) \leqslant y\}$. We have $f(a) \leqslant f(f_*(y)) = f(\bigvee A) = \bigvee f[A] \leqslant y$.

It follows that $f \dashv f_*$. ■

By duality, we have

Theorem 1.1.14. *Let X and Y be complete lattices, and let $Y \xrightarrow{g} X$ be a monotone map. Then g is a right adjoint iff g preserves all meets.*

As for notation, we take note of the following. If $X \xrightarrow{f} Y$ is a join-preserving map between complete lattices, then we denote its right adjoint by f_*, where $f_*(y) = \bigvee\{x \in X \mid f(x) \leqslant y\}$. Dually, if $Y \xrightarrow{g} X$ is a meet-preserving map between complete lattices, then we denote its left adjoint by g^*, where $g^*(x) = \bigwedge\{y \in Y \mid x \leqslant g(y)\}$.

1.1.2 Heyting algebras

We now introduce Heyting algebras, which form part of the centrepiece of locale theory. Our references for this section are [13, 27].

Remark 1.1.15. Throughout the rest of this thesis, all lattices are assumed to be bounded. That is, we assume our lattices to have joins and meets of *all finite* subsets[1]. For example, the chain $(\mathbb{N}, \leqslant)$ (the naturals with the usual order) is not a lattice, since $\bigwedge \varnothing$ does not exist in $\mathbb{N}$ (i.e. $(\mathbb{N}, \leqslant)$ lacks a top element).

Definition 1.1.16. Let A be a lattice. A *Heyting operation* on A is a binary operation of *implication* $\Rightarrow$ which satisfies the following condition:

$$(\forall a, x, y \in A) \qquad a \wedge x \leqslant y \ \text{ iff } \ x \leqslant (a \Rightarrow y). \tag{1.2}$$

The condition (1.2) is known as the *Heyting condition*.

Definition 1.1.17. A (complete) lattice equipped with a Heyting operation is called a (*complete*) *Heyting algebra*.

Remark 1.1.18. For a Heyting algebra A, we often refer to the Heyting operation $\Rightarrow$ as the *Heyting implication*.

Observation 1.1.19. Let A be a Heyting algebra, and fix an $a \in A$. The Heyting condition (1.2) says precisely that the monotone map $a \wedge (-)$ is left adjoint to $a \Rightarrow (-)$. Thus, when A is a complete Heyting algebra, we have

$$(a \Rightarrow y) = \bigvee \{x \in A \mid a \wedge x \leqslant y\}.$$

Furthermore, since right adjoints are unique (when they exist), it follows that a given Heyting algebra has one and only one Heyting operation defined on it (i.e. a lattice admits at most one Heyting operation defined on it).

Proposition 1.1.20. *A complete lattice A is a complete Heyting algebra iff, for each $a \in A$ and $(x_i)_{i \in I} \subseteq A$, we have*

$$a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i). \tag{1.3}$$

Proof. ($\Rightarrow$) Let A be a complete Heyting algebra, and fix an $a \in A$. The Heyting condition (1.2) states that $a \wedge (-)$ is a left adjoint and thus preserves all joins. We thus immediately obtain Eq.(1.3).

($\Leftarrow$) Let A be a complete lattice such that Eq.(1.3) holds for all $a \in A$ and $(x_i)_{i \in I} \subseteq A$. For each $a \in A$, $a \wedge (-)$ preserves all joins and thus has a *unique* right adjoint $a \Rightarrow (-)$. ■

Thus, internally, the important property of a complete Heyting algebra is that the finite meets distribute over the arbitrary joins.

A morphism of Heyting algebras is a morphism of the underlying lattices which preserves the Heyting implication. Similarly, a morphism of complete Heyting algebras is a lattice homomorphism which preserves all joins and the Heyting implication. We therefore obtain two categories: **Hey** (the category of Heyting algebras and their morphisms) and **cHey** (the category of complete Heyting algebras and their morphisms).

[1] It suffices to require that our lattices have 0 and 1, and binary joins and binary meets.

Proposition 1.1.21. *Every Heyting algebra is a distributive lattice.*

Proof. For a Heyting algebra A, and $a \in A$, the monotone map $a \wedge (-)$ preserves all joins. Hence A is a distributive lattice. ∎

Remark 1.1.22. Thus, to construct a lattice which is not Heyting, it suffices to construct a lattice which is not distributive.

Proposition 1.1.23. *Every Boolean algebra is a Heyting algebra.*

Proof. Let A be a Boolean algebra. For all $a, y \in A$, put $a \Rightarrow y := \neg a \vee y$. We claim that $\Rightarrow$ is the Heyting implication on A. We have:

$$\begin{aligned} x \leqslant (a \Rightarrow y) &\iff x \leqslant \neg a \vee y \\ &\implies a \wedge x \leqslant a \wedge (\neg a \vee y) \\ &\iff a \wedge x \leqslant (a \wedge \neg a) \vee (a \wedge y) \\ &\iff a \wedge x \leqslant a \wedge y \\ &\iff a \wedge x \leqslant y. \end{aligned}$$

In the converse direction:

$$\begin{aligned} a \wedge x \leqslant y &\implies \neg a \vee (a \wedge x) \leqslant (a \Rightarrow y) \\ &\iff (\neg a \vee a) \wedge (\neg a \vee x) \leqslant (a \Rightarrow y) \\ &\iff \neg a \vee x \leqslant (a \Rightarrow y) \\ &\iff x \leqslant (a \Rightarrow y). \end{aligned}$$

We can conclude that $\Rightarrow$, as defined above, is the Heyting implication on the Boolean algebra A. ∎

Let A be a Heyting algebra, and fix $a, b \in A$. Recall that $a \Rightarrow b$ is the largest element $x \in A$ for which $a \wedge x \leqslant b$:

$$(\forall x \in A) \quad a \wedge x \leqslant b \ \text{ iff } \ x \leqslant (a \Rightarrow b).$$

Proposition 1.1.24. *For any Heyting algebra A, and $a, b, c \in A$, we have:*

(*i*) $a \Rightarrow a = 1$,

(*ii*) $b \wedge (a \Rightarrow b) = b$,

(*iii*) $a \wedge (a \Rightarrow b) = a \wedge b$,

(*iv*) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$, *and*

(*v*) $a \leqslant b$ *iff* $a \Rightarrow b = 1$.

Proof. (i) $a \Rightarrow a$ is the largest element $x \in A$ for which $a \wedge x \leqslant a$. Thus $a \Rightarrow a = 1$.

(ii) We have $a \wedge b \leqslant b$, so $b \leqslant (a \Rightarrow b)$, i.e. $b \wedge (a \Rightarrow b) = b$.

(iii) Since $(a \Rightarrow b) \leqslant (a \Rightarrow b)$, we have $a \wedge (a \Rightarrow b) \leqslant b$. Also, $(a \Rightarrow b) \leqslant (a \Rightarrow a)$, so $a \wedge (a \Rightarrow b) \leqslant a$. This yields $a \wedge (a \Rightarrow b) \leqslant a \wedge b$. Now $a \wedge (a \wedge b) \leqslant b$, so $a \wedge b \leqslant (a \Rightarrow b)$, and hence $a \wedge b \leqslant a \wedge (a \Rightarrow b)$. We can conclude that $a \wedge (a \Rightarrow b) = a \wedge b$.

(iv) This follows since the map $a \Rightarrow (-) : A \longrightarrow A$ is a right adjoint and thus preserves all existing meets in A.

(v) We have

$$\begin{aligned} a \leqslant b &\text{ iff } a \wedge 1 \leqslant b \wedge 1 \\ &\text{ iff } 1 = a \Rightarrow (b \wedge 1) \\ &\text{ iff } 1 = (a \Rightarrow b) \wedge \underbrace{(a \Rightarrow 1)}_{=1} \\ &\text{ iff } 1 = a \Rightarrow b. \end{aligned}$$

■

1.2 Adjoint functors

Intuitively, adjunction is a special 'tight' relationship that may be exhibited between two functors (in opposing directions). Adjunctions pervade mathematics, and adjoint functors play a central rôle in category theory. The initial definition for adjunction between functors is rather intuitive: it specifies a 'natural' bijection between particular hom-sets. The naturality axioms in this definition imply the existence of two particular natural transformations (the unit and counit), and these satisfy the so-called 'triangular identities'. Conversely, for a given pair of functors in opposing directions, if there exist natural transformations satisfying the triangular identities, then the functors are adjoint. Furthermore, adjoint functors can be characterised in terms of universal properties determined by the unit and counit. Every adjunction restricts to an equivalence between full subcategories in a canonical way, these subcategories are called the *centre* of the adjunction and the objects in the centre are called the fixed points. If the adjunction is idempotent, then there is an elegant characterisation for the fixed points. We will go over the basic theory of adjoint functors, since these play a prominent rôle in locale theory; we will later come across a fundamental adjunction between spaces and locales, as well as free-forgetful adjunctions for frames (and also when working with frames we use Galois connections to define localic maps from frame homomorphisms). We will provide three equivalent ways of defining adjunctions. Our central reference for this section is [21, ch. 2].

1.2.1 Adjunctions in terms of natural bijections

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and consider a pair of functors in opposing directions:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}.$$

Take objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Suppose that there is a bijection between morphisms of the form $FC \longrightarrow D$ in $\mathcal{D}$ and morphisms of the form $C \longrightarrow GD$ in $\mathcal{C}$, i.e. that there is a bijection between the following hom-sets:

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, GD). \tag{1.4}$$

We can display this bijection, an isomorphism of sets, (1.4) as

$$\frac{FC \longrightarrow D}{C \longrightarrow GD}. \tag{1.5}$$

Let us denote the invertible correspondence (1.4) and (1.5) by a horizontal bar, in both directions, for example;

$$(FC \xrightarrow{g} D) \longmapsto (C \xrightarrow{\overline{g}} GD),$$

$$(FC \xrightarrow{\overline{f}} D) \longleftarrow\!\shortmid (C \xrightarrow{f} GD).$$

We call $\overline{f}$ the transpose of f, and $\overline{g}$ the transpose of g. We impose that $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$ (which is possible since the correspondence (1.5) is a bijection), i.e. that the process of transposition is self-inverse. We say that this bijection (1.4) is *natural* in C and D iff the following naturality axioms hold:

(i) $\overline{(FC \xrightarrow{g} D \xrightarrow{q} D')} = (C \xrightarrow{\overline{g}} GD \xrightarrow{Gq} GD')$, i.e. $(\overline{q \cdot g} : C \longrightarrow GD') = (Gq \cdot \overline{g} : C \longrightarrow GD')$, for all such $\mathcal{D}$-morphisms g, q, and

(ii) $\overline{(C' \xrightarrow{p} C \xrightarrow{f} GD)} = (FC' \xrightarrow{Fp} FC \xrightarrow{\overline{f}} D)$, i.e. $(\overline{f \cdot p} : FC' \longrightarrow D) = (\overline{f} \cdot Fp : FC' \longrightarrow D)$, for all such $\mathcal{C}$-morphisms f, p.

Notice that naturality axiom (ii) is the dual of (i), thus, in what follows, we will be able to deploy the principle of duality in our arguments.

Definition 1.2.1. Take categories and functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}.$$

We say that F is *left adjoint to* G (and G is *right adjoint to* F, and write $F \dashv G$) iff for all $C \in \mathcal{C}$ and all $D \in \mathcal{D}$ there is a bijection between hom-sets as in (1.4) which is **natural** in C and D. We say that there is an *adjunction* between F and G to mean that there is a **natural** bijection as in (1.4) for each C and D.

We will often write $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ to mean that there is an adjunction between functors F and G, with F left adjoint to G. In this case we will also say that there is an adjunction between the *categories* $\mathcal{C}$ and $\mathcal{D}$. The rôle of naturality in Definition 1.2.1 may seem obscure, however, we shall see that it simply implies the existence of two particular (and important) natural transformations for any given adjunction, furthermore, it allows us to infer that the notions of left and right adjoint are dual to one another.

Example 1.2.2. Let X and Y be posets. Viewing X and Y as categories, a pair of monotone maps (functors) $\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$ constitutes an adjunction $f \dashv g$ iff

$$(\forall x \in X)(\forall y \in Y)\ Y(f(x), y) \cong X(x, g(y)) \tag{1.6}$$

naturally in x and y. Now, the hom-sets on either side of (1.6) have at most one element, and every diagram in a poset commutes (so the naturality requirements hold trivially), we therefore have

$$f \dashv g \quad \text{iff} \quad (\forall x \in X)(\forall y \in Y)\ f(x) \leqslant y \iff x \leqslant g(y).$$

Therefore, a Galois connection between posets is precisely the same thing as an adjunction between them (viewing the posets as categories).

Important properties of adjoint functors

If a given functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ admits a left adjoint F, then F is unique up to natural isomorphism: we may speak of *the* left adjoint of G. Dually, if a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a right adjoint G, then G is unique up to natural isomorphism.

A useful result about adjoints is that left adjoints preserve colimits, and right adjoints preserve limits. A very particular case of this occurs with Galois connections between posets: recall that left Galois adjoints preserve joins, and right Galois adjoints preserve meets.

Adjunctions can be composed. Suppose we have adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{E}. \arrow[l, bend left, "G'"] \end{tikzcd}$$

We can compose functors to give $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F'F"] & \mathcal{E} \arrow[l, bend left, "GG'"] \end{tikzcd}$. We obtain $F'F \dashv GG'$, as follows.

- For all $C \in \mathcal{C}$ and all $E \in \mathcal{E}$ we have $\mathcal{D}(FC, G'E) \cong \mathcal{C}(C, GG'E)$.
- For all $C \in \mathcal{C}$ and all $E \in \mathcal{E}$ we have $\mathcal{E}(F'FC, E) \cong \mathcal{D}(FC, G'E)$.

Therefore, $\mathcal{E}(F'FC, E) \cong \mathcal{C}(C, GG'E)$ naturally in $C \in \mathcal{C}$ and $E \in \mathcal{E}$.

1.2.2 Adjunctions in terms of units and counits

Suppose we have an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$. For each $C \in \mathcal{C}$ consider the identity morphism on FC, $\mathrm{id}_{FC} : FC \longrightarrow FC$. Now, id_{FC} has a unique transpose $\eta_C := \overline{\mathrm{id}_{FC}} : C \longrightarrow GFC$ – thus $\overline{\eta_C} = \mathrm{id}_{FC}$. We then obtain a family of $\mathcal{C}$-morphisms $\eta := (C \xrightarrow{\eta_C} GFC)_{C \in \mathcal{C}}$.

Proposition 1.2.3. *η is a natural transformation* $\mathrm{id}_{\mathcal{C}} \Longrightarrow GF$.

Proof. Let $C' \xrightarrow{p} C$ be a $\mathcal{C}$-morphism. We need to verify that the square

$$\begin{tikzcd} C' \arrow[r, "p"] \arrow[d, "\eta_{C'}"'] & C \arrow[d, "\eta_C"] \\ GFC' \arrow[r, "GFp"'] & GFC \end{tikzcd} \tag{1.7}$$

commutes. Notice that

$$\begin{aligned} \overline{(C' \xrightarrow{p} C \xrightarrow{\eta_C} GFC)} &= (FC' \xrightarrow{Fp} FC \xrightarrow{\mathrm{id}_{FC}} FC) \\ &= (FC' \xrightarrow{Fp} FC). \end{aligned}$$

That is to say, $\overline{\eta_C \cdot p} = Fp$, and hence $\eta_C \cdot p = \overline{Fp}$. On the other hand,

$$\begin{aligned} &(FC' \xrightarrow{Fp} FC) = (FC' \xrightarrow{\mathrm{id}_{FC'}} FC' \xrightarrow{Fp} FC) \\ \Longrightarrow\ &(C' \xrightarrow{\overline{Fp}} GFC) = (C' \xrightarrow{\eta_{C'}} GFC' \xrightarrow{GFp} GFC). \end{aligned}$$

So we have $\overline{Fp} = GFp \cdot \eta_{C'}$, and we can conclude that the square (1.7) commutes. ∎

Notice that the commutativity of the naturality square (1.7) follows from the naturality axioms in the definition of adjunction. The natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{id}_{\mathcal{C}}"] \arrow[r, bend right, "GF"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C} \end{tikzcd} \quad \text{with components} \quad \begin{tikzcd} C \arrow[d, "\eta_C"] \\ GFC \end{tikzcd}$$

(as defined above) is called the *unit* of the adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$.

Dually, for each $D \in \mathcal{D}$ we can consider the identity morphism on GD, $\mathrm{id}_{GD} : GD \longrightarrow GD$. Taking the transpose of id_{GD}, we define $\varepsilon_D := \overline{\mathrm{id}_{GD}} : FGD \longrightarrow D$, i.e. $\overline{\varepsilon_D} = \mathrm{id}_{GD}$. This defines a family of $\mathcal{D}$-morphisms $\varepsilon := (FGD \xrightarrow{\varepsilon_D} D)_{D \in \mathcal{D}}$. Using the result of Proposition 1.2.3 and the principle of duality, we obtain the following result.

Fact 1.2.4. *ε is a natural transformation $FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$.*

This result (Fact 1.2.4) could have been proved directly in a similar fashion to that of Proposition 1.2.3, however we now emphasise the power of the duality principle, and in future we will always deploy duality whenever possible. The natural transformation

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{D} \end{tikzcd} \quad \text{with components} \quad \begin{tikzcd} FGD \arrow[d, "\varepsilon_D"] \\ D \end{tikzcd}$$

(as defined above) is called the *counit* of the adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$. It should be noted that the unit $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$ and counit $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$ are dual to each other.

Proposition 1.2.5. *Given an adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *with unit η and counit ε, the triangle (in $\mathcal{D}$)*

$$\begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, "\mathrm{id}_{FC}"'] & FGFC \arrow[d, "\varepsilon_{FC}"] \\ & FC \end{tikzcd} \tag{1.8}$$

commutes for all $C \in \mathcal{C}$.

Proof. Take an object $C \in \mathcal{C}$. We have

$$(C \xrightarrow{\eta_C} GFC) = (C \xrightarrow{\eta_C} GFC \xrightarrow{\mathrm{id}_{GFC}} GFC),$$

and taking the transpose of both sides, we obtain

$$(FC \xrightarrow{\mathrm{id}_{FC}} FC) = (FC \xrightarrow{F\eta_C} FGFC \xrightarrow{\varepsilon_{FC}} FC).$$

That is to say, $\mathrm{id}_{FC} = \varepsilon_{FC} \cdot F\eta_C$. ■

Dualising Proposition 1.2.5, we obtain

Fact 1.2.6. *Given an adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *with unit η and counit ε, the triangle (in $\mathcal{C}$)*

$$\begin{tikzcd} GD & GFGD \arrow[l, "G\varepsilon_D"'] \\ & GD \arrow[ul, "\mathrm{id}_{GD}"] \arrow[u, "\eta_{GD}"'] \end{tikzcd} \tag{1.9}$$

commutes for all $D \in \mathcal{D}$.

The commutative triangles (1.8) and (1.9) are known as the *triangular identities.*

Proposition 1.2.7. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *be an adjunction, with unit η and counit ε. For any $\mathcal{D}$-morphism of the form $FC \xrightarrow{g} D$ we have $\overline{g} = Gg \cdot \eta_C$, and for any $\mathcal{C}$-morphism of the form $C \xrightarrow{f} GD$ we have $\overline{f} = \varepsilon_D \cdot Ff$.*

Proof. Consider a $\mathcal{D}$-morphism $FC \xrightarrow{g} D$, then

$$(C \xrightarrow{\overline{g}} GD) = \overline{(FC \xrightarrow{\mathrm{id}_{FC}} FC \xrightarrow{g} D)} = (C \xrightarrow{\eta_C} GFC \xrightarrow{Gg} GD).$$

By duality, given a $\mathcal{C}$-morphism $C \xrightarrow{f} GD$, we have

$$(FC \xrightarrow{\overline{f}} D) = (FC \xrightarrow{Ff} FGD \xrightarrow{\varepsilon_D} D).$$

■

Proposition 1.2.7 tells us the following. Consider an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} \qquad \text{with unit } \eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF \text{ and counit } \varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}.$$

- Given any $\mathcal{D}$-morphism of the form $FC \xrightarrow{g} D$, the triangle

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\overline{g}"'] & GFC \arrow[d, "Gg"] \\ & GD \end{tikzcd}$$

  commutes. We can state this result slightly differently, in terms of a universal property. For any $\mathcal{C}$-morphism of the form $C \xrightarrow{f} GD$ there exists a unique $\mathcal{D}$-morphism $FC \xrightarrow{\overline{f}} D$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & GFC \arrow[d, "G\overline{f}"] \\ & GD \end{tikzcd}$$

  commutes. That is to say, in diagrammatic form,

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\forall f"'] & GFC \arrow[d, "G\overline{f}"] \\ & GD \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, dashed, "\exists! \overline{f}"'] \\ D. \end{tikzcd} \tag{1.10}$$

  The components of the adjunction unit η satisfy the universal property (1.10).

- Dually, for any C-morphism of the form $C \xrightarrow{f} GD$, the triangle

$$
\begin{tikzcd}
D & FGD \arrow[l, "\varepsilon_D"'] \\
 & FC \arrow[ul, "\overline{f}"] \arrow[u, "Ff"']
\end{tikzcd}
$$

commutes. This can be restated in terms of a universal property, satisfied by the components of the adjunction counit ε. Stated in diagrammatic form:

$$
\begin{tikzcd}
D & FGD \arrow[l, "\varepsilon_D"'] & & GD \\
 & FC \arrow[ul, "\forall g"] \arrow[u, "F\overline{g}"'] & & C. \arrow[u, dashed, "\exists! \overline{g}"]
\end{tikzcd}
\tag{1.11}
$$

(1.11) is a universal property of the adjunction $F \dashv G$ that is satisfied by the components of the counit ε.

Theorem 1.2.8. *Take categories and functors* $\begin{tikzcd} C \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$. *There is a bijective correspondence between*:

(i) *adjunctions of the form* $F \dashv G$, *and*

(ii) *pairs* $(\eta : \mathrm{id}_C \Longrightarrow GF,\ \varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}})$ *of natural transformations satisfying the triangular identities.*

Proof. We have already shown that an adjunction of the form $F \dashv G$ gives rise to a pair (η, ε) of natural transformations satisfying the triangular identities. We now have to show that this process is bijective.

Take a pair of natural transformations $(\eta : \mathrm{id}_C \Longrightarrow GF,\ \varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}})$ satisfying the triangular identities. We must show that there exists a unique adjunction $F \dashv G$ with unit η and counit ε. Uniqueness follows from Proposition 1.2.7: for if there is any adjunction $F \dashv G$ with unit η and counit ε, then, for $C \in C$ and $D \in \mathcal{D}$, the natural bijection $\mathcal{D}(FC, D) \cong C(C, GD)$ is *uniquely* determined by the formulae

$$
\begin{aligned}
\mathcal{D}(FC, D) \ni g &\longmapsto \overline{g} = Gg \cdot \eta_C, \\
C(C, GD) \ni f &\longmapsto \overline{f} = \varepsilon_D \cdot Ff.
\end{aligned}
$$

We must thus show the existence of an adjunction $F \dashv G$ with unit η and counit ε. For each $C \in C$ and $D \in \mathcal{D}$, define functions

$$
\begin{tikzcd}
\mathcal{D}(FC, D) \arrow[r, shift left] & C(C, GD) \arrow[l, shift left]
\end{tikzcd}
\tag{1.12}
$$

both denoted by a horizontal bar, as follows.

- Given $FC \xrightarrow{g} D$, put $(C \xrightarrow{\overline{g}} GD) := (C \xrightarrow{\eta_C} GFC \xrightarrow{Gg} GD)$.

- Dually, given $C \xrightarrow{f} GD$, put $(FC \xrightarrow{\overline{f}} D) := (FC \xrightarrow{Ff} FGD \xrightarrow{\varepsilon_D} D)$.

We claim that the thus defined functions $g \longmapsto \overline{g}$ and $f \longmapsto \overline{f}$ are mutually inverse. Consider the triangular identity (which commutes)

$$
\begin{tikzcd}
FC \arrow[r, "F\eta_C"] \arrow[dr, "\mathrm{id}_{FC}"'] & FGFC \arrow[d, "\varepsilon_{FC}"] \\
 & FC
\end{tikzcd}
\tag{1.13}
$$

and, for a $\mathcal{D}$-morphism $FC \xrightarrow{g} D$, the (commutative) naturality square

$$\begin{tikzcd} FGFC \arrow[r, "FGg"] \arrow[d, "\varepsilon_{FC}"'] & FGD \arrow[d, "\varepsilon_D"] \\ FC \arrow[r, "g"'] & D. \end{tikzcd} \tag{1.14}$$

Combining the commutative diagrams (1.13) and (1.14) into one commutative diagram, we obtain

$$\begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, "\mathrm{id}_{FC}"'] & FGFC \arrow[r, "FGg"] \arrow[d, "\varepsilon_{FC}"] & FGD \arrow[d, "\varepsilon_D"] \\ & FC \arrow[r, "g"'] & D. \end{tikzcd} \tag{1.15}$$

From the commutativity of (1.15) we have

$$\begin{aligned} g &= g \cdot \mathrm{id}_{FC} \\ &= \varepsilon_D \cdot FGg \cdot F\eta_C \\ &= \varepsilon_D \cdot F(Gg \cdot \eta_C) \\ &= \varepsilon_D \cdot F\overline{g} \\ &= \overline{\overline{g}}. \end{aligned}$$

Dually, for a C-morphism $C \xrightarrow{f} GD$, we obtain $f = \overline{\overline{f}}$. We thus have the required bijection (1.12). To observe the naturality of the bijection (1.12), consider a composition of $\mathcal{D}$-morphisms of the form $FC \xrightarrow{g} D \xrightarrow{q} D'$. We have $\overline{q \cdot g} = G(q \cdot g) \cdot \eta_C = Gq \cdot (Gg \cdot \eta_C) = Gq \cdot \overline{g}$, giving the first naturality requirement. The second naturality requirement follows dually. We therefore have an adjunction $F \dashv G$. The components of the unit are given by $\overline{\mathrm{id}_{FC}} = G(\mathrm{id}_{FC}) \cdot \eta_C = \mathrm{id}_{GFC} \cdot \eta_C = \eta_C$, i.e. the unit is η. Dually the counit is ε. ■

Corollary 1.2.9. *Take categories and functors* $\begin{tikzcd} C \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$. *If there are natural transformations* $\eta : \mathrm{id}_C \Longrightarrow GF$ *and* $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$ *which satisfy the triangular identities, then* $F \dashv G$. *In this case,* η *is the unit and* ε *the counit of the adjunction* $F \dashv G$.

Theorem 1.2.8 allows us to conclude that an adjunction may be regarded as a quadruple $(F, G, \eta, \varepsilon)$ of functors and natural transformations satisfying the triangular identities.

Example 1.2.10. Let X and Y be posets, and consider monotone maps $\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$. Since every diagram in a poset commutes, Theorem 1.2.8 and Corollary 1.2.9 tell us that $f \dashv g$ iff $\mathrm{id}_X \leqslant gf$ and $fg \leqslant \mathrm{id}_Y$ (the triangular identities are automatically satisfied). Of course, this we already directly proved previously, but it is nonetheless illuminating to see how this result arises from the general theory of adjoint functors.

We now introduce the notion of *idempotent adjunction* (see [25] for details), which is of use in the interaction between spaces and locales.

Definition 1.2.11. Let $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ be an adjunction with unit $\eta : \mathrm{id}_C \Longrightarrow GF$ and counit $\varepsilon : FG \Longrightarrow$

$\mathrm{id}_{\mathcal{D}}$. The adjunction $F \dashv G$ is said to be *idempotent* iff either of the following equivalent conditions hold:

(i) $\eta_{GD} : GD \longrightarrow GFGD$ is an isomorphism for all $D \in \mathcal{D}$.

(ii) $\varepsilon_{FC} : FGFC \longrightarrow FC$ is an isomorphism for all $C \in \mathcal{C}$.

Every adjunction restricts to an equivalence

An equivalence of categories may be regarded as an adjunction for which the unit and counit are natural isomorphisms. In the opposite direction, every adjunction restricts to an equivalence between full subcategories in a canonical way – we will now spell this out.

Suppose we have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} \qquad \text{with unit } \eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF \text{ and counit } \varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}. \tag{1.16}$$

- Write $\mathbf{Fix}(GF)$ for the full subcategory of $\mathcal{C}$ whose objects are those $C \in \mathcal{C}$ for which $\eta_C : C \longrightarrow GFC$ is an isomorphism.

- Dually, write $\mathbf{Fix}(FG)$ for the full subcategory of $\mathcal{D}$ whose objects are those $D \in \mathcal{D}$ for which $\varepsilon_D : FGD \longrightarrow D$ is an isomorphism.

$\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$ are called the *fixed subcategories* (or the *centre*) of the adjunction (1.16). The objects in the centre are known as the *fixed points* (or *fixed objects*) of the adjunction (1.16). We will show that there is an equivalence between the fixed subcategories on either side of the adjunction (1.16).

Proposition 1.2.12. *Consider the adjunction* (1.16). *Given $C \in \mathcal{C}$, if $\eta_C : C \longrightarrow GFC$ is an isomorphism, then $\varepsilon_{FC} : FGFC \longrightarrow FC$ is an isomorphism.*

Proof. Let $\eta_C : C \longrightarrow GFC$ be an isomorphism. Then $F\eta_C : FC \longrightarrow FGFC$ is an isomorphism, since functors preserve isomorphism. Now consider the commutative triangular identity

$$\begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, "\mathrm{id}_{FC}"'] & FGFC \arrow[d, "\varepsilon_{FC}"] \\ & FC. \end{tikzcd}$$

We have

$$\begin{aligned} & \varepsilon_{FC} \cdot F\eta_C = \mathrm{id}_{FC} \\ \Longleftrightarrow\ & \varepsilon_{FC} = \mathrm{id}_{FC} \cdot (F\eta_C)^{-1} \\ \Longleftrightarrow\ & \varepsilon_{FC} = F(\eta_C^{-1}). \end{aligned}$$

Thus ε_{FC} is an isomorphism. ∎

By duality, we have

Fact 1.2.13. *Consider the adjunction* (1.16). *Given $D \in \mathcal{D}$, if $\varepsilon_D : FGD \longrightarrow D$ is an isomorphism, then $\eta_{GD} : GD \longrightarrow GFGD$ is an isomorphism.*

We now obtain an important corollary.

Corollary 1.2.14. *Consider the adjunction* (1.16)*. For any* $C \in \mathbf{Fix}(GF)$*, we have* $FC \in \mathbf{Fix}(FG)$*. Dually, for any* $D \in \mathbf{Fix}(FG)$*, we have* $GD \in \mathbf{Fix}(GF)$*.*

Theorem 1.2.15. *Consider the adjunction* (1.16)*. The category* $\mathbf{Fix}(GF)$ *is equivalent to the category* $\mathbf{Fix}(FG)$*.*

Proof. Define functors $\mathbf{Fix}(GF) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathbf{Fix}(FG)$ by:

$$\begin{cases} \forall C \in \mathbf{Fix}(GF) & F'C := FC, \\ \forall C_1, C_1 \in \mathbf{Fix}(GF), \forall f \in \mathbf{Fix}(GF)(C_1, C_2) & F'f := Ff. \end{cases}$$

$$\begin{cases} \forall D \in \mathbf{Fix}(FG) & G'D := GD, \\ \forall D_1, D_2 \in \mathbf{Fix}(FG), \forall g \in \mathbf{Fix}(FG)(D_1, D_2) & G'g := Gg. \end{cases}$$

Next, consider the natural transformations

$$\begin{tikzcd} \mathbf{Fix}(GF) \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "G'F'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \mathbf{Fix}(GF) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{Fix}(FG) \arrow[r, bend left, "F'G'"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\varepsilon'"] & \mathbf{Fix}(FG), \end{tikzcd}$$

for which their respective components are defined by

$$\forall C \in \mathbf{Fix}(GF) \quad (C \xrightarrow{\eta'_C} G'F'C) := (C \xrightarrow{\eta_C} GFC), \text{ and}$$
$$\forall D \in \mathbf{Fix}(FG) \quad (F'G'D \xrightarrow{\varepsilon'_D} D) := (FGD \xrightarrow{\varepsilon_D} D).$$

η' and ε' satisfy the triangular identities (because η and ε do), hence η' is the unit and ε' the counit of the adjunction $F' \dashv G'$. Since, for all $C \in \mathbf{Fix}(GF)$ and all $D \in \mathbf{Fix}(FG)$, η_C and ε_D are isomorphisms, it follows that η'_C and ε'_D are isomorphisms, i.e. that η' and ε' are natural isomorphisms. Thus $\mathbf{Fix}(GF) \simeq \mathbf{Fix}(FG)$. ∎

Proposition 1.2.16. *If the adjunction* (1.16) *is idempotent, then, for each* $C \in \mathcal{C}$*,* $C \in \mathbf{Fix}(GF)$ *iff* $C \cong GD$ *for some* $D \in \mathcal{D}$*.*

Proof. Let the adjunction (1.16) be idempotent, and take $C \in \mathcal{C}$.

($\Rightarrow$) Suppose $C \in \mathbf{Fix}(GF)$. Then $\eta_C : C \longrightarrow GFC$ is an isomorphism. That is to say, $C \cong GFC$.

($\Leftarrow$) Suppose there is an isomorphism $\varphi : C \longrightarrow GD$ for some $D \in \mathcal{D}$. We obtain a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\eta_C"'] & GD \arrow[d, "\eta_{GD}"] \\ GFC \arrow[r, "GF\varphi"'] & GFGD, \end{tikzcd}$$

i.e. $\eta_{GD} \cdot \varphi = GF\varphi \cdot \eta_C$. The adjunction (1.16) is idempotent, so η_{GD} is an isomorphism, hence we obtain an isomorphism $\eta_C = (GF\varphi)^{-1} \cdot \eta_{GD} \cdot \varphi$. Thus $C \in \mathbf{Fix}(GF)$. ∎

Dualising Proposition 1.2.16 we obtain

Fact 1.2.17. *If the adjunction* (1.16) *is idempotent, then, for each* $D \in \mathcal{D}$*,* $D \in \mathbf{Fix}(FG)$ *iff* $D \cong FC$ *for some* $C \in \mathcal{C}$*.*

1.2.3 Adjunctions in terms of universal properties

Recall that every adjunction $F \dashv G$ has the universal properties (1.10) and (1.11) which are satisfied by the unit η and counit ε respectively. We will now show that, in the opposite direction, if there are natural transformations η and ε which satisfy the universal properties (1.10) and (1.11) respectively, then there is an adjunction $F \dashv G$ having unit η and counit ε. In fact it will turn out that it is enough to know either the unit or the counit satisfying the respective universal property – the entire adjunction follows from there.

Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. Let there be an assignment $\mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$ where, for $C \in \mathrm{ob}(\mathcal{C})$, we write $C \longmapsto FC \in \mathrm{ob}(\mathcal{D})$. Let $(C \xrightarrow{\eta_C} GFC)_{C \in \mathcal{C}}$ be a family of $\mathcal{C}$-morphisms satisfying the following universal property:

Property 1.2.18. For any $\mathcal{C}$-morphism of the form $C \xrightarrow{f} GD$ (where $D \in \mathcal{D}$) there exists a unique $\mathcal{D}$-morphism $FC \xrightarrow{\overline{f}} D$ such that $f = G\overline{f} \cdot \eta_C$, i.e.

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\forall f"'] & GFC \arrow[d, "G\overline{f}"] \\ & GD \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, dashed, "\exists ! \overline{f}"'] \\ D. \end{tikzcd}$$

By Property 1.2.18, for a given $\mathcal{C}$-morphism $C \xrightarrow{p} C'$, there exists a unique $\mathcal{D}$-morphism $FC \xrightarrow{Fp} FC'$ such that the square

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "p"'] & GFC \arrow[d, "GFp"] \\ C' \arrow[r, "\eta_{C'}"'] & GFC' \end{tikzcd} \tag{1.17}$$

commutes. We thus obtain a function $\mathcal{C}(C, C') \longrightarrow \mathcal{D}(FC, FC')$, where $p \longmapsto Fp$ such that Fp is the unique $\mathcal{D}$-morphism associated with p making (1.17) commute.

Proposition 1.2.19. *The assignments $C \longmapsto FC$ and $(C \xrightarrow{p} C') \longmapsto Fp$ define a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, moreover, by setting $\eta := (C \xrightarrow{\eta_C} GFC)_{C \in \mathcal{C}}$ we obtain a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$.*

Proof. Let $C \xrightarrow{p} C' \xrightarrow{q} C''$ be $\mathcal{C}$-morphisms. By Property 1.2.18 there exist unique $\mathcal{D}$-morphisms $(FC \xrightarrow{Fp} FC')$, $(FC' \xrightarrow{Fq} FC'')$ and $(FC \xrightarrow{F(q \cdot p)} FC'')$ making the diagrams

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "p"'] & GFC \arrow[d, "GFp"] \\ C' \arrow[r, "\eta_{C'}"'] \arrow[d, "q"'] & GFC' \arrow[d, "GFq"] \\ C'' \arrow[r, "\eta_{C''}"'] & GFC'' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "q \cdot p"'] & GFC \arrow[d, "GF(q \cdot p)"] \\ C'' \arrow[r, "\eta_{C''}"'] & GFC'' \end{tikzcd}$$

commute. Now, $F(q \cdot p)$ is the *unique* $\mathcal{D}$-morphism satisfying $GF(q \cdot p) \cdot \eta_C = \eta_{C''} \cdot (q \cdot p)$. But we also have $(GFq \cdot GFp) \cdot \eta_C = G(Fq \cdot Fp) \cdot \eta_C = \eta_{C''} \cdot (q \cdot p)$. Hence $F(q \cdot p) = Fq \cdot Fp$. Lastly, consider the identity morphism on an object $C \in \mathcal{C}$, $\mathrm{id}_C : C \longrightarrow C$. $F(\mathrm{id}_C)$ is the unique $\mathcal{D}$-morphism satisfying $GF(\mathrm{id}_C) \cdot \eta_C = \eta_C$. But we also have $\eta_C = G(\mathrm{id}_{FC}) \cdot \eta_C$, hence $F(\mathrm{id}_C) = \mathrm{id}_{FC}$. We can conclude that we have a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. Moreover, by the commutativity of (1.17), we can conclude that $\eta := (C \xrightarrow{\eta_C} GFC)_{C \in \mathcal{C}}$ is a natural transformation $\mathrm{id}_{\mathcal{C}} \Longrightarrow GF$. ∎

The data thus obtained consists of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$ satisfying Property 1.2.18. By Property 1.2.18, for all $D \in \mathcal{D}$, there exists a unique $\mathcal{D}$-morphism $\varepsilon_D : FGD \longrightarrow D$ for which the triangle

$$\begin{tikzcd} GD \arrow[r, "\eta_{GD}"] \arrow[dr, "\mathrm{id}_{GD}"'] & GFGD \arrow[d, "G\varepsilon_D"] \\ & GD \end{tikzcd} \tag{1.18}$$

commutes – compare this to the triangular identity (1.9). This yields a family $\varepsilon := (FGD \xrightarrow{\varepsilon_D} D)_{D \in \mathcal{D}}$ of $\mathcal{D}$-morphisms. Note that we will need to use the identity

$$\mathrm{id}_{GD} = G\varepsilon_D \cdot \eta_{GD}$$

in our arguments that follow.

Proposition 1.2.20. *ε is a natural transformation $FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$.*

Proof. Let $D \xrightarrow{h} D'$ be a $\mathcal{D}$-morphism. We need to verify that the square

$$\begin{tikzcd} FGD \arrow[r, "FGh"] \arrow[d, "\varepsilon_D"'] & FGD' \arrow[d, "\varepsilon_{D'}"] \\ D \arrow[r, "h"'] & D' \end{tikzcd} \tag{1.19}$$

commutes. Consider the $\mathcal{C}$-morphism $GD \xrightarrow{Gh} GD'$. By Property 1.2.18 there exists a unique $\mathcal{D}$-morphism $\overline{Gh} : FGD \longrightarrow D'$ for which $Gh = G(\overline{Gh}) \cdot \eta_{GD}$. We have

$$\begin{aligned} Gh &= Gh \cdot \mathrm{id}_{GD} \\ &= Gh \cdot (G\varepsilon_D \cdot \eta_{GD}) \\ &= G(h \cdot \varepsilon_D) \cdot \eta_{GD}, \end{aligned}$$

that is to say, $\overline{Gh} = h \cdot \varepsilon_D$. Setting $u := G(\varepsilon_{D'} \cdot FGh) \cdot \eta_{GD} = G\varepsilon_{D'} \cdot GFGh \cdot \eta_{GD}$, we would like to show that $u = Gh$. Consider the (commutative) diagram

$$\begin{tikzcd} GD \arrow[r, "Gh"] \arrow[d, "\eta_{GD}"'] & GD' \arrow[d, "\eta_{GD'}"] \arrow[dr, "\mathrm{id}_{GD'}"] & \\ GFGD \arrow[r, "GFGh"'] & GFGD' \arrow[r, "G\varepsilon_{D'}"'] & GD'. \end{tikzcd}$$

We see that

$$\begin{aligned} u &= G\varepsilon_{D'} \cdot GFGh \cdot \eta_{GD} \\ &= \mathrm{id}_{GD'} \cdot Gh \\ &= Gh. \end{aligned}$$

Thus $\varepsilon_{D'} \cdot FGh = \overline{Gh}$, and we can conclude that the square (1.19) commutes. Therefore ε is a natural transformation $FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$. ∎

Lemma 1.2.21. *For all $C \in \mathcal{C}$, the diagram*

$$\begin{tikzcd} FC & FGFC \arrow[l, "\varepsilon_{FC}"'] \\ & FC \arrow[ul, "\mathrm{id}_{FC}"] \arrow[u, "F\eta_C"'] \end{tikzcd} \tag{1.20}$$

commutes.

Proof. Consider the commutative naturality square

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & GFC \arrow[d, "\eta_{GFC}"] \\ GFC \arrow[r, "GF\eta_C"'] & GFGFC. \end{tikzcd}$$

We have

$$\begin{aligned} G\varepsilon_{FC} \cdot GF\eta_C \cdot \eta_C &= \underbrace{G\varepsilon_{FC} \cdot \eta_{GFC}}_{\mathrm{id}_{GFC}} \cdot \eta_C \qquad \text{(recall that } \mathrm{id}_{GD} = G\varepsilon_D \cdot \eta_{GD}) \\ &= \eta_C. \end{aligned}$$

Thus $G(\varepsilon_{FC} \cdot F\eta_C) = G\varepsilon_{FC} \cdot GF\eta_C = \mathrm{id}_{GFC} = G(\mathrm{id}_{FC})$, which implies that $\varepsilon_{FC} \cdot F\eta_C = \mathrm{id}_{FC}$ ∎

Compare the result of Lemma 1.2.21 to the triangular identity (1.8).

Proposition 1.2.22. *For any $\mathcal{D}$-morphism of the form $FC \xrightarrow{g} D$ (where $C \in \mathcal{C}$), there exists a unique $\mathcal{C}$-morphism $C \xrightarrow{\overline{g}} GD$ such that $g = \varepsilon_D \cdot F\overline{g}$, i.e.*

$$\begin{tikzcd} D & FGD \arrow[l, "\varepsilon_D"'] \\ & FC \arrow[ul, "\forall g"] \arrow[u, "F\overline{g}"'] \end{tikzcd} \qquad \begin{tikzcd} GD \\ C. \arrow[u, dashed, "\exists! \overline{g}"] \end{tikzcd}$$

Proof. Given $FC \xrightarrow{g} D$ define $\overline{g} \coloneqq (Gg \cdot \eta_C) : C \longrightarrow GD$, and consider the commutative naturality square

$$\begin{tikzcd} FGFC \arrow[r, "FGg"] \arrow[d, "\varepsilon_{FC}"'] & FGD \arrow[d, "\varepsilon_D"] \\ FC \arrow[r, "g"'] & D. \end{tikzcd}$$

Then

$$\begin{aligned} \varepsilon_D \cdot F\overline{g} &= \varepsilon_D \cdot F(Gg \cdot \eta_C) \\ &= \underbrace{\varepsilon_D \cdot FGg}_{g \cdot \varepsilon_{FC}} \cdot F\eta_C \\ &= g \cdot \underbrace{\varepsilon_{FC} \cdot F\eta_C}_{\mathrm{id}_{FC}} \\ &= g. \end{aligned}$$

Now let $C \xrightarrow{h} GD$ be a $\mathcal{C}$-morphism such that $g = \varepsilon_D \cdot Fh$, and consider the commutative naturality square

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\eta_C"'] & GD \arrow[d, "\eta_{GD}"] \\
GFC \arrow[r, "GFh"'] & GFGD.
\end{tikzcd}$$

We have

$$\begin{aligned}
\overline{g} &= Gg \cdot \eta_C \\
&= G\varepsilon_D \cdot GFh \cdot \eta_C \\
&= G\varepsilon_D \cdot \eta_{GD} \cdot h \\
&= \mathrm{id}_{GD} \cdot h \\
&= h.
\end{aligned}$$

Therefore, $\overline{g} = Gg \cdot \eta_C$ is the unique $\mathcal{C}$-morphism for which $g = \varepsilon_D \cdot F\overline{g}$. ■

What we have shown is that we have two natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$ and $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$ which satisfy the commutative triangles (1.18) and (1.20) (i.e. the triangular identities for an adjunction). We can therefore conclude that $F \dashv G$.

1.2.4 A summary of the characterisations of adjoint functors

The previous argument in Subsection 1.2.3 can be summarised in the following theorem:

Theorem 1.2.23. *A functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a right adjoint iff for each $C \in \mathcal{C}$ there exists an object $FC \in \mathcal{D}$ and a $\mathcal{C}$-morphism $C \xrightarrow{\eta_C} GFC$ with the property that for any $\mathcal{C}$-morphism of the form $C \xrightarrow{f} GD$ (where $D \in \mathcal{D}$) there exists a unique $\mathcal{D}$-morphism $FC \xrightarrow{\overline{f}} D$ such that $f = G\overline{f} \cdot \eta_C$.*

When the requirements of Theorem 1.2.23 are satisfied, the F can be modified into a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ as described in Proposition 1.2.19, we then have $F \dashv G$. Dualising Theorem 1.2.23 we obtain

Theorem 1.2.24. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left adjoint iff for each $D \in \mathcal{D}$ there exists an object $GD \in \mathcal{C}$ and a $\mathcal{D}$-morphism $FGD \xrightarrow{\varepsilon_D} D$ with the property that for any $\mathcal{D}$-morphism of the form $FC \xrightarrow{g} D$ (where $C \in \mathcal{C}$) there exists a unique $\mathcal{C}$-morphism $C \xrightarrow{\overline{g}} GD$ such that $g = \varepsilon_D \cdot F\overline{g}$.*

Collectively, we can summarise our characterisations for adjoint functors as follows.

Theorem 1.2.25. *Consider a pair of functors in opposing directions $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$. The following statements are equivalent:*

(*i*) *$F \dashv G$.*

(*ii*) *There exist natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$ and $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$ satisfying the triangular identities.*

(*iii*) *There exists a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$ with the property that for any $\mathcal{C}$-morphism of the form $C \xrightarrow{f} GD$ (where $D \in \mathcal{D}$) there exists a unique $\mathcal{D}$-morphism $FC \xrightarrow{\overline{f}} D$ such that $f = G\overline{f} \cdot \eta_C$.*

(*iv*) *There exists a natural transformation $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$ with the property that for any $\mathcal{D}$-morphism of the form $FC \xrightarrow{g} D$ (where $C \in \mathcal{C}$) there exists a unique $\mathcal{C}$-morphism $C \xrightarrow{\overline{g}} GD$ such that $g = \varepsilon_D \cdot F\overline{g}$.*

Chapter 2

From spaces to locales: locale theory as an extension of sober space topology

Every topological space possesses a well-behaved lattice of open sets: a so-called *open-set lattice*. Certain morphisms of open-set lattices are induced by continuous maps f by taking preimage along f. It turns out that for most everyday topological spaces (the sober ones) all possible topological information is carried by the open-set lattice, hence we do not lose anything by studying these open-set lattices directly. Open-set lattices may be abstracted to objects called *frames* which are complete lattices satisfying an infinite distributive law. Similarly, morphisms of open-set lattices are abstracted to *frame homomorphisms*. The passage from the category of topological spaces to the category of frames is *contravariant* [14], therefore the category of locales is formally introduced as the opposite of the category of frames. The category of locales is a covariant 'extension' of the category of sober spaces: the category of sober spaces is embedded in the category of locales as a full subcategory. In passing from spaces to locales, we forget about the points (a point of a locale is a secondary notion, and indeed many locales have no points at all [32]), it is rather the frame of opens that becomes the primary notion. We bear in mind the following sentiment by P.T. Johnstone (see [15]):

"*the topological structure of a locale cannot "live on its points"; from here, it is a small step to make the topological structure (i.e. the frame of open subsets) into the extensional essence of the locale (so that the points, if any, live on the open sets rather than the other way about)*."

A good example which nicely displays the theory of adjunctions is provided through the interplay between spaces and locales. It is well-known that every space yields a locale via its open-set lattice. In the opposite direction, every locale gives rise to a space through the locale's set of points equipped with a natural topology. Both of these processes can be made functorial; these functors are adjoint, moreover, the adjunction is idempotent – meaning that the fixed points on either side are the sober spaces and spatial locales, which together form the centre of the adjunction (i.e. that the category of sober spaces is equivalent to the category of spatial locales). The main references used in this chapter are [13, 17, 32].

2.1 Introducing frames and locales

Let X be a topological space, and write $\mathcal{O}X$ for its poset of open subsets ordered by inclusion. Since $\mathcal{O}X$ is closed under arbitrary unions, it follows that $\mathcal{O}X$ is a complete lattice: for $(U_i)_{i\in I} \subseteq \mathcal{O}X$ we have

$$\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i \quad \text{and} \quad \bigwedge_{i\in I} U_i = \mathrm{int}\left(\bigcap_{i\in I} U_i\right),$$

where int is the interior operator on X. We call $\mathcal{O}X$ the *open-set lattice* of X. Furthermore, we have the following distributive law in $\mathcal{O}X$:

$$V \wedge \bigvee_{i\in I} U_i = \bigvee_{i\in I} (V \wedge U_i) \quad \text{for all } V \in \mathcal{O}X \text{ and all } (U_i)_{i\in I} \subseteq \mathcal{O}X. \tag{2.1}$$

That Eq. (2.1) holds is because finite meet in $\mathcal{O}X$ coincides with finite set-theoretic intersection, which is well-known to distribute over arbitrary union. The order-dual of Eq. (2.1) will in general not hold in $\mathcal{O}X$ since arbitrary meet in $\mathcal{O}X$ coincides with the interior of the intersection, thus we cannot expect finite join (finite union) to distribute over arbitrary meet. For an example to see how the order-dual of Eq. (2.1) can fail in an open-set lattice, see [30].

For a continuous map $X \xrightarrow{f} Y$ of spaces, we obtain a map $\mathcal{O}Y \xrightarrow{f^*} \mathcal{O}X$ between open-set lattices, where

$$f^*(U) \coloneqq f^{-1}[U].$$

Such a map f^* of open-set lattices always preserves arbitrary joins and finite meets (since the preimage function preserves all unions and intersections). We cannot expect f^* to preserve arbitrary meets because of the difficulty which arises with the interior operator. Abstracting the essential properties of such open-set lattices $\mathcal{O}X$ and maps f^*, we have

Definition 2.1.1. A *frame* is a complete lattice A which satisfies the following infinite distributive law:

$$a \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I} (a \wedge x_i) \tag{2.2}$$

for all $a \in A$ and all $(x_i)_{i\in I} \subseteq A$. A *frame homomorphism* $h : A \longrightarrow B$ is a function between frames which preserves arbitrary joins (including the bottom 0) and finite meets (including the top 1). We refer to Eq. (2.2) as the *frame distributivity property*. A frame homomorphism is often referred to as a *frame map*.

Originally, before the term *frame* was introduced by Hugh Dowker and Dona Papert-Strauss in September 1964 (see [7]), such lattices were referred to as *local lattices* as was coined by Charles Ehresmann and his student Jean Bénabou around the year 1957 (see [14, 16]). Generally, the accepted terminology for such lattices is to call them frames. A *coframe* is a complete lattice which satisfies the order-dual of Eq. (2.2); as we alluded to earlier, a frame is not necessarily a coframe (dually, a coframe is not necessarily a frame).

We note that topological properties (invariants) intuitively only involve statements about open sets: topological properties 'live' on the open-set lattices. Thus it would seem natural to study the open-set lattices (or abstractions thereof, i.e. frames) directly.

Since every topological space X yields a frame $\mathcal{O}X$, and every continuous map $f : X \longrightarrow Y$ yields a frame homomorphism $f^* : \mathcal{O}Y \longrightarrow \mathcal{O}X$, we would like to view frames as a lattice-theoretic generalisation of topological spaces. Our goal is therefore to construct a theory where we can 'do' topology on a lattice-theoretic foundation. We will write **Frm** for the category of frames and frame homomorphisms. Writing **Top** for the category of topological spaces and continuous maps, we see that there is an obvious functor $\mathcal{O} : \mathbf{Top}^{\mathrm{op}} \longrightarrow \mathbf{Frm}$, where $X \longmapsto \mathcal{O}X$ and $(X \xrightarrow{f} Y) \longmapsto (\mathcal{O}Y \xrightarrow{f^*} \mathcal{O}X)$. We will later see that not all frames are of the form $\mathcal{O}X$ for a topological space X. We choose to view $\mathcal{O}$ as a functor $\mathbf{Top} \longrightarrow \mathbf{Frm}^{\mathrm{op}}$, since we already know how to 'do' topology in **Top**, and we want to somehow extend the theory of topological spaces to a theory of frames. It should be noted that in any complete lattice A we always have $\bigvee_{i\in I}(a \wedge x_i) \leqslant a \wedge \bigvee_{i\in I} x_i$ for all $a \in A$ and all $(x_i)_{i\in I} \subseteq A$. We thus have

Fact 2.1.2. *A complete lattice A is a frame iff*

$$a \wedge \bigvee_{i\in I} x_i \leqslant \bigvee_{i\in I}(a \wedge x_i)$$

for all $a \in A$ and all $(x_i)_{i\in I} \subseteq A$.

Observation 2.1.3. By recalling Proposition 1.1.20, we see that a lattice is a frame iff it is a complete Heyting algebra.

We may thus use the terms frame and complete Heyting algebra interchangeably (frames and complete Heyting algebras are extensionally the same thing) – as long as no reference is made to morphisms: in general frame homomorphisms do not preserve the Heyting implication (see [17]). For a frame A, the Heyting implication is given by the following useful formula:

$$(a \Rightarrow y) = \bigvee\{x \in A \mid a \wedge x \leqslant y\}.$$

We remark that **cHey** is a non-full subcategory of **Frm**. Since every Boolean algebra is Heyting, it follows that every complete Boolean algebra is a frame. In the context of frames, we will refer to complete Boolean algebras as *Boolean frames*. We thus have some concrete examples of frames:

- complete Boolean algebras,
- any topology on a set,
- finite distributive lattices (since all joins are finite), and
- complete chains (since any chain is a Heyting algebra).

We should point out that since frame homomorphisms preserve arbitrary joins, they have uniquely defined right adjoints. Such right adjoints will be referred to as *localic maps*.

Definition 2.1.4. A function $f : A \longrightarrow B$ between frames is said to be a *localic map* iff f preserves arbitrary meets and its left adjoint f^* preserves finite meets.

Notationally, we will generally denote frame homomorphisms by the letter h (so that h's right adjoint h_* is a localic map) and localic maps by the letter f (so that f's left adjoint f^* is a frame homomorphism).

Observation 2.1.5. Every continuous map $X \xrightarrow{f} Y$ of spaces gives rise to a localic map $\mathcal{O}X \xrightarrow{f_*} \mathcal{O}Y$, where $f^* \dashv f_*$.

Proposition 2.1.6. *Let $X \xrightarrow{f} Y$ be a continuous map of spaces. Then, for all $U \in \mathcal{O}X$,*

$$f_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]}.$$

Proof. Take $U \in \mathcal{O}X$. We have

$$\begin{aligned} f_*(U) &= \bigcup\{V \in \mathcal{O}Y \mid f^*(V) \subseteq U\} \\ &= \bigcup\{V \in \mathcal{O}Y \mid f^{-1}[V] \subseteq U\}. \end{aligned}$$

Now,

$$\begin{aligned} f^{-1}[V] \subseteq U &\iff X \smallsetminus U \subseteq X \smallsetminus f^{-1}[V] \\ &\iff X \smallsetminus U \subseteq f^{-1}[Y \smallsetminus V] \\ &\iff f[X \smallsetminus U] \subseteq Y \smallsetminus V. \end{aligned}$$

Therefore,

$$\begin{aligned} f_*(U) &= \bigcup\{V \in \mathcal{O}Y \mid f[X \smallsetminus U] \subseteq Y \smallsetminus V\} \\ &= Y \smallsetminus \bigcap\{Y \smallsetminus V \mid V \in \mathcal{O}Y \text{ and } f[X \smallsetminus U] \subseteq Y \smallsetminus V\} \\ &= Y \smallsetminus \bigcap\{A \subseteq Y \mid A \text{ closed and } f[X \smallsetminus U] \subseteq A\} \\ &= Y \smallsetminus \overline{f[X \smallsetminus U]}. \end{aligned}$$

■

We have the following characterisation for localic maps, but first we need a proposition.

Proposition 2.1.7. *Let* $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "g"] \end{tikzcd}$ *be an adjoint pair of monotone maps between Heyting algebras. Then f preserves finite meets iff, for each $a \in A$ and $b \in B$,*

$$g(f(a) \Rightarrow b) = a \Rightarrow g(b). \tag{2.3}$$

Proof. ($\Rightarrow$) Let f preserve finite meets. Take $a, a' \in A$ and $b \in B$. Then

$$\begin{aligned} a' \leqslant g(f(a) \Rightarrow b) &\text{ iff } f(a') \leqslant f(a) \Rightarrow b \\ &\text{ iff } f(a) \wedge f(a') \leqslant b \\ &\text{ iff } f(a \wedge a') \leqslant b \\ &\text{ iff } a \wedge a' \leqslant g(b) \\ &\text{ iff } a' \leqslant a \Rightarrow g(b). \end{aligned}$$

($\Leftarrow$) Suppose that, for all $a \in A$ and $b \in B$, $g(f(a) \Rightarrow b) = a \Rightarrow g(b)$. Take $a, a' \in A$ and $b \in B$. Then

$$\begin{aligned} f(a) \wedge f(a') \leqslant b &\text{ iff } f(a') \leqslant f(a) \Rightarrow b \\ &\text{ iff } a' \leqslant g(f(a) \Rightarrow b) \\ &\text{ iff } a' \leqslant a \Rightarrow g(b) \\ &\text{ iff } a \wedge a' \leqslant g(b) \\ &\text{ iff } f(a \wedge a') \leqslant b. \end{aligned}$$

■

Eq. (2.3) is known as the *Frobenius identity*. We thus obtain the Frobenius characterisation for localic maps:

Corollary 2.1.8. *A function $f : A \longrightarrow B$ between frames which preserves all meets is a localic map iff, for each $a \in A$ and $b \in B$,*

$$f(f^*(b) \Rightarrow a) = b \Rightarrow f(a).$$

Recall that we have a functor $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Frm}^{\mathrm{op}}$. Thus, the passage from the category of topological spaces to the category of frames is contravariant [14]. One of the goals of point-free topology is to build a theory where we can perform topological methods on a lattice-theoretic foundation. Therefore, if we wish to somehow extend the theory of topological spaces into a theory of frames, we have to take seriously this contravariance: it is not the category **Frm** in which we would like to 'do' topology, but rather in its opposite $\mathbf{Frm}^{\mathrm{op}}$. We note that $\mathbf{Frm}^{\mathrm{op}}$ is an abstract category: the morphisms in $\mathbf{Frm}^{\mathrm{op}}$ are *not* 'structure-preserving' functions between frames. We therefore introduce new terminology:

Definition 2.1.9. A *locale* is extensionally the same object as a frame, except a *locale morphism* $f : A \longrightarrow B$ is defined by a frame homomorphism $f^* : B \longrightarrow A$. We write **Loc** for the category of locales and locale morphisms, that is, $\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}}$.

The functor $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Frm}^{\mathrm{op}}$ is now regarded as a functor $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Loc}$; a space X is mapped to its open-set locale $\mathcal{O}X$, and a continuous map $f : X \longrightarrow Y$ is mapped to the locale morphism $\widetilde{f} : \mathcal{O}X \longrightarrow \mathcal{O}Y$ defined by the frame homomorphism $f^{-1} = f^* : \mathcal{O}Y \longrightarrow \mathcal{O}X$. We call $\mathcal{O}$ the *open-set functor*. As for notation, given a morphism of locales $f : A \longrightarrow B$, we will write $f^* : B \longrightarrow A$ for the frame homomorphism that defines f, and $f_* : A \longrightarrow B$ for the right adjoint of f^* (f_* is of course a localic map).

Remark 2.1.10. We now have three names for describing the same type of object (extensionally): A is a frame iff A is a locale iff A is a complete Heyting algebra. The difference is in the morphisms! The categories **Frm**, **Loc**, and **cHey** are all distinct from each other.

2.2 Points

All nonempty topological spaces contain points, and the open sets are defined by the points (the open sets 'live' on the points). The elements of a locale A can be thought of as 'open sets' – these are the primary notion of the locale. However, if we wish to view locales as a type of *space*, we should also have the notion of point. The problem is that we do not foundationally have available the notion of point of the locale A. This calls for the *categorification* of the set-theoretic notion of point.

Let 1 denote the terminal object in **Top**. That is, $1 = \{*\}$ with opens $\mathcal{O}1 = \{\emptyset, 1\}$. As frames, $\mathcal{O}1 \cong 2 := \{0, 1\}$ (2 is the two-element Boolean algebra). It is easily seen that 2 is the initial frame: for any frame A there is precisely one frame homomorphism $!_A : 2 \longrightarrow A$ defined by $0 \longmapsto 0_A$ and $1 \longmapsto 1_A$. Thus, 2 is the terminal object in **Loc**. **Frm** also has a terminal object $1 = \{*\}$ which corresponds to the empty topological space.

Let X be a topological space. A point $p \in X$ is precisely identified with a continuous map $\overline{p} : 1 \longrightarrow X$ where $\overline{p}(*) = p$. The assignment $p \longmapsto \overline{p}$ defines a bijection. This observation leads to the following definition:

Definition 2.2.1. A *point* of a locale (frame) A is a locale morphism $f : 2 \longrightarrow A$. Equivalently, it is a frame homomorphism $p : A \longrightarrow 2$. We write $\mathrm{pt}(A)$ for the set of all points of the locale (frame) A, that is, $\mathrm{pt}(A) = \{p \mid p \text{ is a frame map } A \longrightarrow 2\} = \mathbf{Frm}(A, 2)$.

2.3 An excursion: the alternative descriptions of points

We will need some results regarding ideals and filters in lattices (for this section the main reference used is [13]). Recall that a *meet-semilattice* is a poset A with all finite meets (in particular, A has a top element 1). Dually, a *join-semilattice* is a poset A with all finite joins (in particular, A has a bottom element 0).

Definition 2.3.1. A subset J of a join-semilattice A is said to be an *ideal* iff:

(i) J is a sub-join-semilattice of A; i.e. $0 \in J$, and $[a, a' \in J \implies a \vee a' \in J]$, and

(ii) J is a down-set; i.e. $(a \in J \text{ and } a' \leqslant a) \implies a' \in J$.

Dualising Definition 2.3.1, we have

Definition 2.3.2. A subset F of a meet-semilattice A is said to be a *filter* iff:

(i) F is a sub-meet-semilattice of A; i.e. $1 \in F$, and $[a, a' \in F \implies a \wedge a' \in F]$, and

(ii) F is an up-set; i.e. $(a \in F \text{ and } a \leqslant a') \implies a' \in F$.

Examples 2.3.3. Here are some examples of ideals and filters:

- If A is a join-semilattice, and $a \in A$, then $\downarrow a := \{x \in A \mid x \leqslant a\}$ is an ideal – it is the *principal ideal* generated by a. When A is finite, all ideals are principal.
- If X is a set, and $x \in X$, then $\mathcal{E}_x = \{A \subseteq X \mid x \notin A\}$ is an ideal in $\mathcal{P}(X)$. The set $\mathcal{F} = \{A \subseteq X \mid A \text{ is finite}\}$ is also an ideal in $\mathcal{P}(X)$. If X is infinite, then the ideal $\mathcal{F}$ is non-principal.
- If $f : A \longrightarrow B$ is a homomorphism of join-semilattices, then $\ker(f) := f^{-1}(0) = \{a \in A \mid f(a) = 0\}$ is an ideal in A. $\ker(f)$ is called the *kernel* of f.
- If A is a meet-semilattice, and $a \in A$, then $\uparrow a := \{x \in A \mid a \leqslant x\}$ is a filter – it is the *principal filter* generated by a. When A is finite, all filters are principal.
- If X is a topological space, and $x \in X$, then $\mathcal{N}_x := \{N \subseteq X \mid N \text{ is a neighbourhood of } x\}$ is a filter in $\mathcal{P}(X)$. $\mathcal{N}_x$ is called the *neighbourhood filter* of x.
- If X is a set, then the set $\{A \subseteq X \mid X \setminus A \text{ is finite}\}$ is a filter in $\mathcal{P}(X)$.
- If $f : A \longrightarrow B$ is a homomorphism of meet-semilattices, then $\mathrm{coker}(f) := f^{-1}(1) = \{a \in A \mid f(a) = 1\}$ is a filter in A. $\mathrm{coker}(f)$ is called the *cokernel* of f.

Definition 2.3.4. Let A be a meet-semilattice. An element $p \in A$ is said to be *prime* iff:

(i) $p \neq 1$, and

(ii) $a \wedge a' = p \implies a = p$ or $a' = p$.

In a finite meet-semilattice it is easy to identify prime elements:

Remark 2.3.5. An element in a finite meet-semilattice is prime iff it has exactly one cover.

For example, consider the Hasse diagram of the 8-element Boolean algebra below; the three shaded vertices are the prime elements – notice that their complements are the atoms of the Boolean algebra.

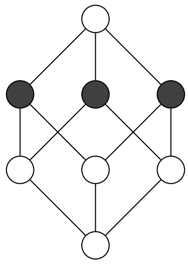

Figure 2.1: A Hasse diagram of the 8-element Boolean algebra (equivalently, the powerset algebra of a 3-element set) in which the prime elements are shaded. The complements of the prime elements are precisely the atoms of the algebra.

Proposition 2.3.6. *For a distributive lattice A, an element $p \in A$ with $p \neq 1$ is prime iff*

$$(\forall a, a' \in A) \quad a \wedge a' \leqslant p \implies a \leqslant p \ \text{ or } \ a' \leqslant p.$$

Proof. ($\Rightarrow$) Let p be prime. Suppose $a \wedge a' \leqslant p$. Then $p \vee (a \wedge a') = (p \vee a) \wedge (p \vee a') = p$, so $p \vee a = p$ or $p \vee a' = p$ (since p is prime), and hence $a \leqslant p$ or $a' \leqslant p$.

($\Leftarrow$) Assume that $(\forall a, a' \in A)$ $a \wedge a' \leqslant p \implies a \leqslant p$ or $a' \leqslant p$. Let $a \wedge a' = p$. Then $p \leqslant a$ and $p \leqslant a'$. Since $a \wedge a' \leqslant p$, it follows that $a \leqslant p$ or $a' \leqslant p$. Therefore $a = p$ or $a' = p$, and hence p is prime. ∎

Definition 2.3.7. Let A be a lattice. An ideal $J \subseteq A$ is said to be *prime* iff:

(i) $1 \notin J$, and

(ii) $a \wedge a' \in J \implies a \in J$ or $a' \in J$.

Proposition 2.3.8. *Let A be a lattice, and let $p \in A$. If the principal ideal ${\downarrow} p$ is prime, then p is prime.*

Proof. Let ${\downarrow} p$ be prime. Then $p \neq 1$. Let $a, a' \in A$ be such that $a \wedge a' = p$. Then $a \wedge a' \in {\downarrow} p$, so $a \in {\downarrow} p$ or $a' \in {\downarrow} p$ (i.e. $a \leqslant p$ or $a' \leqslant p$). Since $a \wedge a' = p$, we have $p \leqslant a$ and $p \leqslant a'$. Thus $a = p$ or $a' = p$. ∎

Proposition 2.3.9. *If the lattice A in Proposition 2.3.8 is distributive, then the converse of Proposition 2.3.8 holds.*

Proof. Let A be distributive, and let $p \in A$ be prime. Suppose $a \wedge a' \in \downarrow p$. Then $a \wedge a' \leqslant p$. Thus $a \leqslant p$ or $a' \leqslant p$ (i.e. $a \in \downarrow p$ or $a' \in \downarrow p$). ■

Remark 2.3.10. Thus, in a distributive lattice, the prime elements are precisely those elements which generate the prime principal ideals.

Definition 2.3.11. A filter F in a lattice A is said to be *prime* iff:

(i) $0 \notin F$, and

(ii) $a \vee a' \in F \implies a \in F$ or $a' \in F$.

Theorem 2.3.12. *Let J be an ideal of a lattice A. The following conditions are equivalent*:

(i) *$A \setminus J$ is a filter.*

(ii) *J is prime.*

(iii) *There exists a lattice homomorphism $f : A \longrightarrow 2$ such that $J = \ker(f)$.*

Proof.

$(i) \Rightarrow (ii)$: Let $A \setminus J$ be a filter. Then $1 \in A \setminus J$, so $1 \notin J$. Let $a, a' \in A$ be such that $a \wedge a' \in J$. Then $a \wedge a' \notin A \setminus J$, so $a \notin A \setminus J$ or $a' \notin A \setminus J$ (because $A \setminus J$ is a filter). Hence $a \in J$ or $a' \in J$. That is, J is prime.

$(ii) \Rightarrow (iii)$: Let J be prime. Define a function $f : A \longrightarrow 2$ by

$$f(a) := \begin{cases} 0 & \text{for } a \in J, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that f is a lattice homomorphism. Since $0 \in J$ and $1 \notin J$, we have $f(0) = 0$ and $f(1) = 1$. Let $a, a' \in A$.

- Suppose $a \wedge a' \in J$. Then $a \in J$ or $a' \in J$ (since J is prime). Hence $f(a \wedge a') = 0 = f(a) \wedge f(a')$.
- Suppose $a \wedge a' \notin J$. Then, for all $x \in A$ with $a \wedge a' \leqslant x$, we have $x \notin J$ (since J is a down-set). In particular, $a \notin J$ and $a' \notin J$. Hence $f(a \wedge a') = 1 = f(a) \wedge f(a')$.
- Suppose $a \vee a' \in J$. Then $a \in J$ and $a' \in J$ (since J is a down-set). Hence $f(a \vee a') = f(a) \vee f(a')$.
- Suppose $a \vee a' \notin J$. Then $a \notin J$ or $a' \notin J$ (since J is an ideal). Hence $f(a) \vee f(a') = 1 = f(a \vee a')$.

It follows that f is a lattice homomorphism, and $J = \ker(f)$.

$(iii) \Rightarrow (i)$: Let $f : A \longrightarrow 2$ be a lattice homomorphism such that $J = \ker(f)$. Then $A \setminus J = \operatorname{coker}(f)$ which is a filter in A.

■

Dualising Theorem 2.3.12, we obtain

Theorem 2.3.13. *Let F be a filter in a lattice A. The following conditions are equivalent*:

(*i*) *$A \setminus F$ is an ideal.*

(*ii*) *F is prime.*

(*iii*) *There exists a lattice homomorphism $f : A \longrightarrow 2$ such that $F = \operatorname{coker}(f)$.*

Corollary 2.3.14. *Let J be a subset of a lattice A. J is a prime ideal iff $A \setminus J$ is a prime filter.*

Proof. ($\Rightarrow$) Let J be a prime ideal. Then $A \setminus J$ is a filter. Since $A \setminus (A \setminus J) = J$ – which is an ideal – it must be that $A \setminus J$ is a *prime* filter.

($\Leftarrow$) Let $A \setminus J$ be a prime filter. Then $A \setminus (A \setminus J) = J$ is an ideal. Moreover, J is a *prime* ideal since $A \setminus J$ is a filter. ■

Definition 2.3.15. Let A be a complete lattice. A prime filter $F \subseteq A$ is said to be *completely prime* iff

$$(\forall M \subseteq A) \ \ \bigvee M \in F \implies (\exists m \in M) \ \ m \in F.$$

Remark 2.3.16. For brevity's sake, instead of saying "completely prime filter", we will say "*cp-filter*".

Proposition 2.3.17. *Let A be a complete lattice, and let $J \subseteq A$. J is a prime principal ideal iff $A \setminus J$ is a cp-filter.*

Proof. ($\Rightarrow$) Let J be a prime principal ideal. Then $J = \downarrow p$ for some $p \in A$ such that $\downarrow p$ is a prime ideal. We must show that $A \setminus (\downarrow p)$ is a cp-filter. By Corollary 2.3.14, $A \setminus (\downarrow p)$ is a prime filter. We have $A \setminus (\downarrow p) = \{a \in A \mid a \nleqslant p\}$. Let $(x_i)_{i \in I} \subseteq A$, and suppose that $\bigvee_{i \in I} x_i \in A \setminus (\downarrow p)$ (i.e. $\bigvee_{i \in I} x_i \nleqslant p$). If $(\forall i \in I)\ x_i \notin A \setminus (\downarrow p)$, then $(\forall i \in I)\ x_i \leqslant p$, which would imply $\bigvee_{i \in I} x_i \leqslant p$ – a contradiction.

($\Leftarrow$) Let $A \setminus J$ be a cp-filter. By Corollary 2.3.14, J is a prime ideal. We must show that $\bigvee J \in J$. Suppose that $\bigvee J \notin J$. Then $\bigvee J \in A \setminus J$. Hence there exists $a \in J$ for which $a \in A \setminus J$ – which is an obvious contradiction. Therefore it must be the case that $\bigvee J \in J$, and hence J is a prime principal ideal. ■

Theorem 2.3.18. *Let A be a frame. Let J be an ideal of A. The following conditions are equivalent*:

(*i*) *$A \setminus J$ is a cp-filter.*

(*ii*) *J is a prime principal ideal.*

(*iii*) *There exists a frame homomorphism $h : A \longrightarrow 2$ such that $J = \ker(h)$.*

Proof. By Proposition 2.3.17, we already know that $(i) \Leftrightarrow (ii)$.

$(ii) \Rightarrow (iii)$: Let J be a prime principal ideal. Then there exists a prime element $p \in A$ such that $J = \downarrow p$ (since A is a distributive lattice, p being prime makes $\downarrow p$ a prime ideal). Define a function $h : A \longrightarrow 2$ by

$$h(a) := \begin{cases} 0 & \text{for } a \leqslant p, \\ 1 & \text{otherwise.} \end{cases}$$

By the proof of Theorem 2.3.12, we know that $h(0) = 0$, $h(1) = 1$, and that h preserves binary meets and binary joins. We thus need to show that h preserves arbitrary joins. Let $(x_i)_{i \in I} \subseteq A$.

- Suppose $\bigvee_{i \in I} x_i \leqslant p$. Then, for all $i \in I$, $x_i \leqslant p$ (i.e. $h(x_i) = 0$). Hence $0 = h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} h(x_i)$.
- Suppose $\bigvee_{i \in I} x_i \nleqslant p$. Then $h(\bigvee_{i \in I} x_i) = 1$, and there exists some $j \in I$ for which $x_j \nleqslant p$ (i.e. $h(x_j) = 1$). Hence $1 = \bigvee_{i \in I} h(x_i) = h(\bigvee_{i \in I} x_i)$.

It follows that h is a frame homomorphism, and $J = \ker(h)$.

$(iii) \Rightarrow (ii)$: Let $h : A \longrightarrow 2$ be a frame homomorphism such that $J = \ker(h)$. Then it is easily verified that J is a prime ideal; we must show that it is principal. That is, we must show that $\bigvee J \in J$. We have

$$\begin{aligned} h(\bigvee J) &= \bigvee h[J] \\ &= \bigvee \{h(a) \mid a \in J\} \\ &= \bigvee \{0\} \\ &= 0. \end{aligned}$$

So $\bigvee J \in J$. We can conclude that J is a prime principal ideal.

■

It is then straightforward to observe that

Corollary 2.3.19. *For any frame A, there is a bijection between each of the following*:

(i) *the frame homomorphisms $A \longrightarrow 2$,*

(ii) *the prime principal ideals of A,*

(iii) *the cp-filters of A, and*

(iv) *the prime elements of A.*

Proof. It follows immediately from Theorem 2.3.18, although, for a detailed proof, see my Honours thesis [30]. ■

2.4 The adjunction between spaces and locales

Now that we have exhibited a functor $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Loc}$, it is natural to ask whether this is an adjoint functor. We shall show that $\mathcal{O}$ is indeed a left adjoint functor. This adjunction between **Top** and **Loc** has a number of pleasant consequences that we shall exhibit in the proceeding section.

Definition 2.4.1. A *subframe* of a frame A is a subset $A' \subseteq A$ which is closed under arbitrary joins and finite meets in A.

Observation 2.4.2. For any set X, a topology on X is just a subframe of the powerset frame $\mathcal{P}(X)$.

Lemma 2.4.3. *Let $h : A \longrightarrow B$ be a frame homomorphism. If A' is a subframe of A, then $h[A']$ is a subframe of B.*

Proof. Let A' be a subframe of A. Since $0_A, 1_A \in A'$, it immediately follows that $0_B, 1_B \in h[A']$. Let $M \subseteq h[A']$. Then there exists $N \subseteq A'$ such that $M = h[N]$. We have $\bigvee h[N] = h(\bigvee N)$, and $\bigvee N \in A'$ (since A' is a subframe). So $h(\bigvee N) = \bigvee M \in h[A']$. It follows that $h[A']$ is closed under arbitrary joins. Let $b, b' \in h[A']$. Then $b = h(a)$ and $b' = h(a')$ for some $a, a' \in A'$. We have $b \wedge b' = h(a) \wedge h(a') = h(a \wedge a')$, and, since $a \wedge a' \in A'$, $b \wedge b \in h[A']$. It follows that $h[A']$ is closed under binary meets. We can conclude that $h[A']$ is a subframe of B. ■

2.4.1 Describing the counit

Let A be a frame, and consider the points $\mathrm{pt}(A)$ (= $\mathbf{Frm}(A, 2)$). Define a function $\varphi_A : A \longrightarrow \mathcal{P}(\mathrm{pt}(A))$ by

$$\varphi_A(a) \coloneqq \{p \in \mathrm{pt}(A) \mid p(a) = 1\}.$$

Proposition 2.4.4. *φ_A is a frame homomorphism.*

Proof. It is straightforward to verify that $\varphi_A(0) = \varnothing$ and $\varphi_A(1) = \mathrm{pt}(A)$. Take $(a_i)_{i \in I} \subseteq A$. We have

$$\begin{aligned}
\varphi_A(\bigvee_{i \in I} a_i) &= \{p \in \mathrm{pt}(A) \mid p(\bigvee_{i \in I} a_i) = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid \bigvee_{i \in I} p(a_i) = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid (\exists i \in I)\ p(a_i) = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid (\exists i \in I)\ p \in \varphi_A(a_i)\} \\
&= \bigcup_{i \in I} \varphi_A(a_i).
\end{aligned}$$

It follows that φ_A preserves arbitrary joins. Now take $a, a' \in A$. We have

$$\begin{aligned}
\varphi_A(a \wedge a') &= \{p \in \mathrm{pt}(A) \mid p(a \wedge a') = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid p(a) \wedge p(a') = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid p(a) = 1 \text{ and } p(a') = 1\} \\
&= \{p \in \mathrm{pt}(A) \mid p \in \varphi_A(a) \text{ and } p \in \varphi_A(a')\} \\
&= \varphi_A(a) \cap \varphi_A(a').
\end{aligned}$$

It follows that φ_A preserves finite meets. We can conclude that φ_A is a frame homomorphism. ■

Corollary 2.4.5. *For any frame A, $\varphi_A[A]$ is a topology on* $\mathrm{pt}(A)$.

Remark 2.4.6. We therefore regard $\mathrm{pt}(A)$ as a topological space equipped with the topology $\mathcal{O}(\mathrm{pt}(A)) \coloneqq \varphi_A[A]$. The space $\mathrm{pt}(A)$ is referred to as the *spectrum* of A. We now view φ_A as a frame homomorphism $\varphi_A : A \longrightarrow \mathcal{O}(\mathrm{pt}(A))$ – in this way φ_A is always surjective.

Let $f : A \longrightarrow B$ be a locale morphism, and let $p \in \mathrm{pt}(A)$. Then we have frame homomorphisms

$$\begin{tikzcd} B \arrow[r, "f^*"] \arrow[rr, bend right, "p \cdot f^*"'] & A \arrow[r, "p"] & 2. \end{tikzcd}$$

Hence $p \cdot f^* \in \mathrm{pt}(B)$, and we write $\mathrm{pt}(f)(p) \coloneqq p \cdot f^*$. We therefore obtain a function $\mathrm{pt}(f) : \mathrm{pt}(A) \longrightarrow \mathrm{pt}(B)$.

Proposition 2.4.7. *For any locale morphism $f : A \longrightarrow B$, the function $\mathrm{pt}(f) : \mathrm{pt}(A) \longrightarrow \mathrm{pt}(B)$ is continuous.*

Proof. Let $b \in B$, and consider the open set $\varphi_B(b) = \{p \in \mathrm{pt}(B) \mid p(b) = 1\}$. We have

$$\begin{aligned} \mathrm{pt}(f)^{-1}[\varphi_B(b)] &= \{p \in \mathrm{pt}(A) \mid \mathrm{pt}(f)(p) \in \varphi_B(b)\} \\ &= \{p \in \mathrm{pt}(A) \mid p \cdot f^* \in \varphi_B(b)\} \\ &= \{p \in \mathrm{pt}(A) \mid p(f^*(b)) = 1\} \\ &= \varphi_A(f^*(b)) \in \mathcal{O}(\mathrm{pt}(A)). \end{aligned}$$

We can conclude that $\mathrm{pt}(f)$ is continuous. ■

Straightforward verification reveals that we have a functor $\mathrm{pt} : \mathbf{Loc} \longrightarrow \mathbf{Top}$, we refer to pt as the *spectrum functor*. We thus have a means of functorially moving between **Top** and **Loc**:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\mathcal{O}"] & \mathbf{Loc} \arrow[l, bend left, "\mathrm{pt}"] \end{tikzcd}.$$

For any locale A, we write $\varepsilon_A : \mathcal{O}(\mathrm{pt}(A)) \longrightarrow A$ for the locale morphism defined by the frame homomorphism $\varphi_A : A \longrightarrow \mathcal{O}(\mathrm{pt}(A))$. That is $\varepsilon_A^* = \varphi_A$. Set $\varepsilon := (\mathcal{O}(\mathrm{pt}(A)) \xrightarrow{\varepsilon_A} A)_{A \in \mathbf{Loc}}$.

Proposition 2.4.8. *ε is a natural transformation* $\begin{tikzcd} \mathbf{Loc} \arrow[r, bend left, "\mathcal{O}\cdot\mathrm{pt}"{name=U}] \arrow[r, bend right, "\mathrm{id}_{\mathbf{Loc}}"'{name=D}] & \mathbf{Loc} \arrow[from=U, to=D, Rightarrow, "\varepsilon"] \end{tikzcd}$.

Proof. Let $f : A \longrightarrow B$ be a locale morphism. We need to verify that the square

$$\begin{tikzcd} \mathcal{O}(\mathrm{pt}(A)) \arrow[r, "\widetilde{\mathrm{pt}(f)}"] \arrow[d, "\varepsilon_A"'] & \mathcal{O}(\mathrm{pt}(B)) \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. By duality, we must verify that the square (in **Frm**)

$$\begin{tikzcd} \mathcal{O}(\mathrm{pt}(A)) & \mathcal{O}(\mathrm{pt}(B)) \arrow[l, "\mathrm{pt}(f)^*"'] \\ A \arrow[u, "\varphi_A"] & B \arrow[l, "f^*"] \arrow[u, "\varphi_B"'] \end{tikzcd} \tag{2.4}$$

commutes. Let $b \in B$. By the proof of Proposition 2.4.7,

$$\begin{aligned} \mathrm{pt}(f)^*(\varphi_B(b)) &= \mathrm{pt}(f)^{-1}[\varphi_B(b)] \\ &= \varphi_A(f^*(b)). \end{aligned}$$

Therefore the square (2.4) commutes, and we can conclude that ε is a natural transformation $\mathcal{O} \cdot \mathrm{pt} \Rightarrow \mathrm{id}_{\mathbf{Loc}}$. ■

2.4.2 Describing the unit

Let X be a topological space. Define a function $\eta_X : X \longrightarrow \mathrm{pt}(\mathcal{O}X)$ by $\eta_X(x) := x^*$, where $x^* : \mathcal{O}X \longrightarrow 2$ is the frame map defined by

$$x^*(U) := \begin{cases} 1 & \text{for } x \in U, \\ 0 & \text{for } x \notin U. \end{cases}$$

(It is straightforward to check that such x^* defines a frame homomorphism $\mathcal{O}X \longrightarrow 2$).

Proposition 2.4.9. *η_X is continuous.*

Proof. Let $U \in \mathcal{O}X$, and consider the open set $\varphi_{\mathcal{O}X}(U) = \{p \in \mathrm{pt}(\mathcal{O}X) \mid p(U) = 1\}$. We have

$$\begin{aligned}\eta_X^{-1}[\varphi_{\mathcal{O}X}(U)] &= \{x \in X \mid x^* \in \varphi_{\mathcal{O}X}(U)\}\\ &= \{x \in X \mid x^*(U) = 1\}\\ &= \{x \in X \mid x \in U\}\\ &= U.\end{aligned}$$

We can conclude that η_X is continuous. ∎

Proposition 2.4.10. *If η_X is surjective, then η_X is an open map.*

Proof. Let η_X be surjective. Let $U \in \mathcal{O}X$. We have

$$\begin{aligned}\eta_X[U] &= \{x^* \mid x \in U\}\\ &= \{x^* \mid x \in X \text{ and } x^*(U) = 1\}\\ &\subseteq \varphi_{\mathcal{O}X}(U).\end{aligned}$$

Let $p \in \varphi_{\mathcal{O}X}(U)$. Then p is a frame map $\mathcal{O}X \longrightarrow 2$ for which $p(U) = 1$. Since η_X is surjective, there exists an $x \in X$ for which $x^* = p$. This x is obviously such that $x \in U$, and hence $p \in \eta_X[U]$. That is, $\varphi_{\mathcal{O}X}(U) \subseteq \eta_X[U]$. It follows that $\eta_X[U] = \varphi_{\mathcal{O}X}(U)$, and we conclude that η_X is an open map. ∎

Next, set $\eta := (X \xrightarrow{\eta_X} \mathrm{pt}(\mathcal{O}X))_{X \in \mathbf{Top}}$.

Proposition 2.4.11. *η is a natural transformation* $\begin{tikzcd}\mathbf{Top} \arrow[r, bend left, "\mathrm{id}_{\mathbf{Top}}"] \arrow[r, bend right, "\mathrm{pt}\cdot\mathcal{O}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Top}\end{tikzcd}$.

Proof. Let $f : X \longrightarrow Y$ be a continuous map of spaces. We need to verify that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \mathrm{pt}(\mathcal{O}X) \arrow[r, "\mathrm{pt}(\widetilde{f})"'] & \mathrm{pt}(\mathcal{O}Y) \end{tikzcd} \tag{2.5}$$

commutes. Let $x \in X$. We have $\eta_Y(f(x)) = (f(x))^*$, where $(f(x))^*$ is the frame map $\mathcal{O}Y \longrightarrow 2$ defined by

$$(f(x))^*(U) := \begin{cases} 1 & \text{for } f(x) \in U,\\ 0 & \text{for } f(x) \notin U.\end{cases}$$

Additionally, $\mathrm{pt}(\widetilde{f})(\eta_X(x)) = \mathrm{pt}(\widetilde{f})(x^*) = x^* \cdot f^*$. For $U \in \mathcal{O}Y$, we have

$$\begin{aligned}x^* \cdot f^*(U) &= x^*(f^{-1}[U])\\ &= \begin{cases} 1 & \text{for } x \in f^{-1}[U],\\ 0 & \text{for } x \notin f^{-1}[U]\end{cases}\\ &= \begin{cases} 1 & \text{for } f(x) \in U,\\ 0 & \text{for } f(x) \notin U.\end{cases}\end{aligned}$$

Therefore the square (2.5) commutes, and we can conclude that η is a natural transformation $\mathrm{id}_{\mathbf{Top}} \Rightarrow \mathrm{pt}\cdot\mathcal{O}$. ∎

2.4.3 Exhibiting the adjunction

So far we have functors $\mathbf{Top} \underset{\mathrm{pt}}{\overset{O}{\rightleftarrows}} \mathbf{Loc}$, and natural transformations $\mathrm{id}_{\mathbf{Top}} \xRightarrow{\eta} \mathrm{pt} \cdot O$ and $O \cdot \mathrm{pt} \xRightarrow{\varepsilon} \mathrm{id}_{\mathbf{Loc}}$. It should thus come as no surprise that

Theorem 2.4.12. *The open-set functor O is left adjoint to the spectrum functor* pt; *this adjunction has unit η and counit ε. Moreover the adjunction is idempotent.*

Proof. We need to verify that the triangular identities for an adjunction are satisfied. Then we can check idempotency.

1. First triangular identity.

Let X be a topological space. We must show that the triangle (in **Loc**)

$$\begin{tikzcd} OX \arrow[r, "\widetilde{\eta_X}"] \arrow[dr, "\mathrm{id}_{OX}"'] & O(\mathrm{pt}(OX)) \arrow[d, "\varepsilon_{OX}"] \\ & OX \end{tikzcd} \tag{2.6}$$

commutes. By duality, we must show that the triangle (in **Frm**)

$$\begin{tikzcd} OX & O(\mathrm{pt}(OX)) \arrow[l, "\eta_X^*"'] \\ & OX \arrow[ul, "\mathrm{id}_{OX}"] \arrow[u, "\varphi_{OX}"'] \end{tikzcd}$$

commutes. Let $U \in OX$. By the proof of Proposition 2.4.9, we have

$$\eta_X^*(\varphi_{OX}(U)) = \eta_X^{-1}[\varphi_{OX}(U)] = U.$$

Thus $\eta_X^* \cdot \varphi_{OX} = \mathrm{id}_{OX}$, and it follows that the triangle (2.6) commutes.

2. Second triangular identity.

Let A be a frame. We must show that the triangle

$$\begin{tikzcd} \mathrm{pt}(A) & \mathrm{pt}(O(\mathrm{pt}(A))) \arrow[l, "\mathrm{pt}(\varepsilon_A)"'] \\ & \mathrm{pt}(A) \arrow[ul, "\mathrm{id}_{\mathrm{pt}(A)}"] \arrow[u, "\eta_{\mathrm{pt}(A)}"'] \end{tikzcd} \tag{2.7}$$

commutes. Let $p \in \mathrm{pt}(A)$. We have $\mathrm{pt}(\varepsilon_A)(\eta_{\mathrm{pt}(A)}(p)) = \mathrm{pt}(\varepsilon_A)(p^*) = p^* \cdot \varphi_A$. Now p^* is the frame homomorphism $O(\mathrm{pt}(A)) \longrightarrow 2$ given by (for $a \in A$)

$$\begin{aligned} p^*(\varphi_A(a)) &= \begin{cases} 1 & \text{for } p \in \varphi_A(a), \\ 0 & \text{for } p \notin \varphi_A(a) \end{cases} \\ &= \begin{cases} 1 & \text{for } p(a) = 1, \\ 0 & \text{for } p(a) = 0 \end{cases} \\ &= p(a). \end{aligned}$$

Thus, $p^* \cdot \varphi_A = p$. It follows that the triangle (2.7) commutes. We can conclude that O is left adjoint to pt; with η the unit and ε the counit of the adjunction.

3. Idempotency.

Since, for any frame A, the homomorphism $\varphi_A : A \longrightarrow O(\mathrm{pt}(A))$ is surjective, it follows that φ_A is an isomorphism iff φ_A is injective. Let X be a topological space. We claim that $\varphi_{OX} : OX \longrightarrow O(\mathrm{pt}(OX))$ is injective. Let $U, V \in OX$. Suppose that $\varphi_{OX}(U) = \varphi_{OX}(V)$. Then, for all $p \in \mathrm{pt}(OX)$, $p(U) = 1$ iff $p(V) = 1$. Let $x \in U$. Then we have the frame map $x^* : OX \longrightarrow 2$ with $x^*(U) = 1$, hence $x^*(V) = 1$, and so $x \in V$. Now instead suppose $x \in V$. Then $x^*(V) = 1$, hence $x^*(U) = 1$, and so $x \in U$. Thus $U = V$, i.e. the frame homomorphism $\varphi_{OX} : OX \longrightarrow O(\mathrm{pt}(OX))$ is injective. It therefore follows that, for all $X \in \mathbf{Top}$, $\varepsilon_{OX} : O(\mathrm{pt}(OX)) \longrightarrow OX$ is an isomorphism in $\mathbf{Loc}$, and hence, for all $A \in \mathbf{Frm}$, $\eta_{\mathrm{pt}(A)} : \mathrm{pt}(A) \longrightarrow \mathrm{pt}(O(\mathrm{pt}(A)))$ is a homeomorphism. We can conclude that the adjunction $O \dashv \mathrm{pt}$ is idempotent.

■

We now know that we have an adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "O"] \arrow[r, phantom, "\perp"] & \mathbf{Loc}. \arrow[l, bend left, "\mathrm{pt}"] \end{tikzcd}$$

Since every adjunction restricts to an equivalence between full subcategories in a canonical way, it is worth investigating the equivalence that arises from the adjunction $O \dashv \mathrm{pt}$.

Definition 2.4.13. A topological space X is said to be *sober* iff $\eta_X : X \longrightarrow \mathrm{pt}(OX)$ is a homeomorphism. We write $\mathbf{Sob}$ for the full subcategory of $\mathbf{Top}$ whose objects are the sober spaces; that is, $\mathbf{Sob} := \mathbf{Fix}(\mathrm{pt}{\cdot}O)$.

Remark 2.4.14. By the idempotency of the adjunction $O \dashv \mathrm{pt}$, a space X is sober iff there exists a frame A for which $X \cong \mathrm{pt}(A)$. Moreover, if (for a given space X) η_X is bijective, then η_X is an open map (recall Proposition 2.4.10); therefore a space X is sober iff η_X is a bijection. For any space X, we refer to $\mathrm{pt}(OX)$ as the *sobrification* of X. Any space X can be replaced by its sobrification $\mathrm{pt}(OX)$ without altering the underlying open-set locale OX, this is because the adjunction $O \dashv \mathrm{pt}$ is idempotent: recall that $\varphi_{OX} : OX \longrightarrow O(\mathrm{pt}(OX))$ is always an isomorphism.

We will soon see that $\mathbf{Sob}$ is a substantial part of $\mathbf{Top}$; in fact every Hausdorff space is sober, and every sober space is T_0. These facts will be proven in Section 2.6.

Definition 2.4.15. A frame (locale) A is said to be *spatial* (or is said to *have enough points*) iff $\varphi_A : A \longrightarrow O(\mathrm{pt}(A))$ is an isomorphism. We write $\mathbf{SpLoc}$ for the full subcategory of $\mathbf{Loc}$ whose objects are the spatial locales ($\mathbf{SpFrm}$ is the full subcategory of $\mathbf{Frm}$ whose objects are the spatial frames). Notice that $\mathbf{SpLoc} := \mathbf{Fix}(O \cdot \mathrm{pt})$.

Remark 2.4.16. By the idempotency of the adjunction $O \dashv \mathrm{pt}$, a frame A is spatial iff there exits a space X for which $A \cong OX$. Since $\varphi_A : A \longrightarrow O(\mathrm{pt}(A))$ is always surjective, a frame A is spatial iff φ_A is injective. For any frame A, we refer to $O(\mathrm{pt}(A))$ as the *spatialisation* of A.

We therefore see that the fixed points on either side of the adjunction $O \dashv \mathrm{pt}$ are the sober spaces and the spatial locales. One thus obtains

Corollary 2.4.17. *The category* $\mathbf{Sob}$ *is equivalent to the category* $\mathbf{SpLoc}$*. In other words,* $\mathbf{Sob}$ *is dual to* $\mathbf{SpFrm}$.

Corollary 2.4.17 is the reason why we refer to the study of frames and locales as *point-free topology*: **Sob** is embedded in **Loc** as a full subcategory – and in passing from **Sob** (or more generally, **Top**) to **Loc** it is the points in spaces which are forgotten and the open sets which are remembered – thus locale theory 'extends' sober space topology. In Section 2.5 we will see that many frames and locales contain no points at all (in particular we will demonstrate that any complete Boolean algebra without atoms contains no points). If A is an atomless complete Boolean algebra (viewed as a frame) then $\mathrm{pt}(A) = \varnothing$, however the cardinality of A is obviously infinite (any finite Boolean algebra is atomic = finite powerset algebra), so A has many "open sets" and no points (A is inherently non-spatial) – thus illustrating that the open sets are more fundamental than the points in locale theory. We will meet a concrete example of this type of non-spatial frame in Chapter 3.

2.5 More on spatiality

Let A be a frame. We write $\mathrm{pr}(A)$ for the set of prime elements in A. By Theorem 2.3.18 and Corollary 2.3.19, $\mathrm{pt}(A)$ is in bijection with $\mathrm{pr}(A)$: the function $\alpha_A : \mathrm{pt}(A) \longrightarrow \mathrm{pr}(A)$ defined by $(\forall p \in \mathrm{pt}(A))\, \alpha_A(p) := \bigvee \ker(p)$ is bijective. (Recall that $\ker(p)$ is a prime principal ideal and hence $\bigvee \ker(p)$ is a prime element in A). Fix an $a \in A$ and $p \in \mathrm{pt}(A)$. We have

$$p \in \varphi_A(a) \iff p(a) = 1 \iff a \notin \ker(p) \iff a \nleq \bigvee \ker(p) \iff a \nleq \alpha_A(p). \tag{2.8}$$

Motivated by the fact that α_A is bijective, and the equivalence (2.8), we are led to define a function $\varphi'_A : A \longrightarrow \mathcal{P}(\mathrm{pr}(A))$ by

$$\varphi'_A(a) := \{x \in \mathrm{pr}(A) \mid a \nleq x\}.$$

In this way we have

$$(\forall a \in A)(\forall p \in \mathrm{pt}(A)) \qquad p \in \varphi_A(a) \iff \alpha_A(p) \in \varphi'_A(a).$$

Remark 2.5.1. In a manner similar to the proof of that of Proposition 2.4.4, one can show that φ'_A is a frame homomorphism.

Proposition 2.5.2. *φ_A is injective iff φ'_A is injective.*

Proof. ($\Rightarrow$) Let φ_A be injective. Let $a, b \in A$ and suppose that $\varphi'_A(a) = \varphi'_A(b)$. Then:

$$\begin{aligned} &(\forall x \in \mathrm{pr}(A)) \qquad x \in \varphi'_A(a) \iff x \in \varphi'_A(b) \\ \text{iff } &(\forall x \in \mathrm{pr}(A)) \qquad \alpha_A^{-1}(x) \in \varphi_A(a) \iff \alpha_A^{-1}(x) \in \varphi_A(b). \end{aligned} \tag{2.9}$$

Since α_A^{-1} is bijective, in particular surjective, by (2.9) above it follows that $\varphi_A(a) = \varphi_A(b)$. Hence $a = b$. Therefore φ'_A is injective.

($\Leftarrow$) Let φ'_A be injective. Let $a, b \in A$ and suppose that $\varphi_A(a) = \varphi_A(b)$. Then:

$$\begin{aligned} &(\forall p \in \mathrm{pt}(A)) \qquad p \in \varphi_A(a) \iff p \in \varphi_A(b) \\ \text{iff } &(\forall p \in \mathrm{pt}(A)) \qquad \alpha_A(p) \in \varphi'_A(a) \iff \alpha_A(p) \in \varphi'_A(b). \end{aligned}$$

Since α_A is bijective, in particular surjective, it follows that $\varphi'_A(a) = \varphi'_A(b)$. Hence $a = b$. Therefore φ_A is injective. ∎

Corollary 2.5.3. *A frame A is spatial iff φ'_A is injective.*

Spatiality has an internal lattice-theoretic characterisation, and it turns out to be quite useful, as we now present below:

Theorem 2.5.4. *A frame A is spatial iff, for each $a \in A$, $a = \bigwedge\{x \in \mathrm{pr}(A) \mid a \leqslant x\}$.*

Proof. ($\Rightarrow$) Let A be spatial. Then φ'_A is injective. Let $a \in A$, and set $P_a := \{x \in \mathrm{pr}(A) \mid a \leqslant x\}$. Clearly $a \leqslant \bigwedge P_a$. Since φ'_A is order-preserving (recall Remark 2.5.1), we have $\varphi'_A(a) \subseteq \varphi'_A(\bigwedge P_a)$. Let $x \in \varphi'_A(\bigwedge P_a)$. Then $\bigwedge P_a \nleqslant x$, hence $x \notin P_a$, and so $a \nleqslant x$. This yields $x \in \varphi'_A(a)$. Therefore $\varphi'_A(\bigwedge P_a) \subseteq \varphi'_A(a)$, and hence $\varphi'_A(a) = \varphi'_A(\bigwedge P_a)$. It then follows that $a = \bigwedge P_a$.

($\Leftarrow$) Suppose that, for all $a \in A$, $a = \bigwedge\{x \in \mathrm{pr}(A) \mid a \leqslant x\}$. Let $a, b \in A$ be such that $a \neq b$. Then $\bigwedge\{x \in \mathrm{pr}(A) \mid a \leqslant x\} \neq \bigwedge\{x \in \mathrm{pr}(A) \mid b \leqslant x\}$, and so $\{x \in \mathrm{pr}(A) \mid a \leqslant x\} \neq \{x \in \mathrm{pr}(A) \mid b \leqslant x\}$. Then there exists $x \in \mathrm{pr}(A)$ such that $a \leqslant x$ and $b \nleqslant x$ (i.e. $x \notin \varphi'_A(a)$ and $x \in \varphi'_A(b)$), or there exists $y \in \mathrm{pr}(A)$ such that $a \nleqslant y$ and $b \leqslant y$ (i.e. $y \in \varphi'_A(a)$ and $y \notin \varphi'_A(b)$). Hence $\varphi'_A(a) \neq \varphi'_A(b)$. So φ'_A is injective, and A is spatial. ∎

We summarise the various useful characterisations of spatiality in the following theorem:

Theorem 2.5.5. *For any frame A, the following conditions are equivalent*:

(i) *The frame map $\varphi_A : A \longrightarrow O(\mathrm{pt}(A))$ is an isomorphism.*

(ii) *The frame map $\varphi_A : A \longrightarrow O(\mathrm{pt}(A))$ is injective.*

(iii) *There exists a topological space X for which $A \cong OX$.*

(iv) *For all $a \in A$, $a = \bigwedge\{x \in \mathrm{pr}(A) \mid a \leqslant x\}$.*

In [6] the notion of the *ascending chain condition* (briefly, the *ACC*) for posets was introduced. The ACC is in essence a generalisation of finiteness for posets; every finite poset satisfies the ACC, but not conversely. In my Honours thesis ([30]) it was shown that, as a corollary of Theorem 2.5.4,

Corollary 2.5.6. *Every frame which satisfies the ACC is spatial.*

In particular, if one wishes to construct a non-spatial frame, such a frame ought to be infinite.

2.5.1 Spatial Boolean frames

Proposition 2.5.7. *In a Boolean algebra A, an element is an atom iff its complement is prime.*

Proof. ($\Rightarrow$) Let a be an atom in A. Consider $\neg a$. Since $a \neq 0$, it follows that $\neg a \neq 1$. Let $x, y \in A$ be such that $x \wedge y = \neg a$. Then $a = \neg(x \wedge y) = \neg x \vee \neg y$. So $\neg x \leqslant a$ and $\neg y \leqslant a$. Since a is an atom, if $\neg x \neq 0$ then $\neg x = a$, and if $\neg y \neq 0$ then $\neg y = a$. Since $\neg x \vee \neg y \neq 0$, it must be that $\neg x \neq 0$ or $\neg y \neq 0$, hence $a = \neg x$ or $a = \neg y$. So $\neg a = x$ or $\neg a = y$. Therefore $\neg a$ is prime.

($\Leftarrow$) Let x be prime in A. Consider $\neg x$. Since $x \neq 1$, it follows that $\neg x \neq 0$. Let $y \in A$ be such that $y \neq 0$ and $y \leqslant \neg x$. Then $x \leqslant \neg y \neq 1$. We have $0 = y \wedge \neg y \leqslant x$. Since x is prime, $y \leqslant x$ or $\neg y \leqslant x$. If $y \leqslant x$, then $\neg x \leqslant \neg y$, so $y \leqslant \neg y$ and hence $y \wedge \neg y = y \neq 0$ – a contradiction. Therefore it must be the case that $y \nleqslant x$ and $\neg y \leqslant x$. Hence we can conclude that $x = \neg y$, i.e. $\neg x = y$. Therefore $\neg x$ is an atom. ∎

Remark 2.5.8. For a Boolean algebra A, we write $\mathrm{at}(A)$ for the set of atoms in A.

Lemma 2.5.9. *A Boolean frame A is spatial iff A is atomic.*

PROOF. Let A be a Boolean frame. A is spatial iff

$$\begin{aligned}
&(\forall a \in A) \quad a = \bigwedge\{x \in \mathrm{pr}(A) \mid a \leqslant x\} \\
\text{iff } &(\forall a \in A) \quad a = \bigwedge\{x \in \mathrm{pr}(A) \mid \neg x \leqslant \neg a\} \\
\text{iff } &(\forall a \in A) \quad a = \bigwedge\{\neg\neg x \mid x \in \mathrm{pr}(A) \text{ and } \neg x \leqslant \neg a\} \\
\text{iff } &(\forall a \in A) \quad a = \neg\bigvee\{\neg x \mid x \in \mathrm{pr}(A) \text{ and } \neg x \leqslant \neg a\} \\
\text{iff } &(\forall a \in A) \quad \neg a = \bigvee\{\neg x \mid x \in \mathrm{pr}(A) \text{ and } \neg x \leqslant \neg a\} \\
\text{iff } &(\forall b \in A) \quad b = \bigvee\{y \in \mathrm{at}(A) \mid y \leqslant b\} \\
\text{iff } &A \text{ is atomic.}
\end{aligned}$$

■

As a consequence of Lemma 2.5.9 we see that the spatial Boolean frames correspond to discrete spaces, moreover,

Corollary 2.5.10. *Any atomless Boolean frame is non-spatial.*

2.6 More on sobriety

Let X be a topological space. We have a bijection $\alpha_{OX} : \mathrm{pt}(OX) \longrightarrow \mathrm{pr}(OX)$, where $\alpha_{OX}(p) = \bigcup \ker(p) = \bigcup\{U \in OX \mid p(U) = 0\}$.

Proposition 2.6.1. *For each $x \in X$, $X \setminus \overline{\{x\}}$ is prime in OX.*

PROOF. Let $x \in X$, and consider the frame map $\eta_X(x) = x^* : OX \longrightarrow 2$. We have

$$\begin{aligned}
\alpha_{OX}(x^*) &= \bigcup \ker(x^*) \\
&= \bigcup\{U \in OX \mid x^*(U) = 0\} \\
&= \bigcup\{U \in OX \mid x \notin U\} \\
&= \bigcup\{X \setminus (X \setminus U) \mid U \in OX \text{ and } x \in X \setminus U\} \\
&= X \setminus \bigcap\{X \setminus U \mid U \in OX \text{ and } x \in X \setminus U\} \\
&= X \setminus \bigcap\{A \subseteq X \mid A \text{ is closed and } x \in A\} \\
&= X \setminus \overline{\{x\}}.
\end{aligned}$$

Thus $X \setminus \overline{\{x\}}$ is prime in OX. ■

If X is sober, then the continuous map $\eta_X : X \longrightarrow \mathrm{pt}(OX)$ is bijective, hence the composite $\alpha_{OX} \cdot \eta_X : X \longrightarrow \mathrm{pr}(X)$ is a bijection. Conversely, suppose $\alpha_{OX} \cdot \eta_X$ is a bijection. Then (since α_{OX} is a bijection) $\eta_X = \alpha_{OX}^{-1} \cdot (\alpha_{OX} \cdot \eta_X)$ and hence η_X is a bijection (i.e. X is sober). Thus we have established

Proposition 2.6.2. *X is sober iff the function $\alpha_{OX} \cdot \eta_X : X \longrightarrow \mathrm{pr}(OX)$ is a bijection.*

Consequently, combining Propositions 2.6.1 and 2.6.2,

Corollary 2.6.3. *X is sober iff*

$$(\forall P \in \mathrm{pr}(\mathcal{O}X))(\exists! x \in X) \quad P = X \setminus \overline{\{x\}}.$$

Let A be a frame. We write $\mathrm{cp}(A)$ for the set of all cp-filters of A. By Theorem 2.3.18 and Corollary 2.3.19, the points of A are in bijection with the cp-filters of A: the function $\beta_A : \mathrm{pt}(A) \longrightarrow \mathrm{cp}(A)$ defined by $\beta_A(p) := \mathrm{coker}(p) = p^{-1}(1)$ is bijective.

Proposition 2.6.4. *Let X be a topological space. For any $x \in X$, the set $\mathcal{U}_x := \{U \in \mathcal{O}X \mid x \in U\}$ is a cp-filter of $\mathcal{O}X$.*

Proof. Take $x \in X$. Then $\eta_X(x) = x^*$ is a point of $\mathcal{O}X$. Hence we have a cp-filter

$$\begin{aligned}\beta_{\mathcal{O}X}(\eta_X(x)) = \mathrm{coker}(x^*) &= \{U \in \mathcal{O}X \mid x^*(U) = 1\} \\ &= \{U \in \mathcal{O}X \mid x \in U\}.\end{aligned}$$

■

If X is sober, then the composite $\beta_{\mathcal{O}X} \cdot \eta_X : X \longrightarrow \mathrm{cp}(\mathcal{O}X)$ is a bijection. Conversely, if $\beta_{\mathcal{O}X} \cdot \eta_X$ is a bijection, then $\eta_X = \beta_{\mathcal{O}X}^{-1} \cdot (\beta_{\mathcal{O}X} \cdot \eta_X)$ and hence η_X is bijective (i.e. X is sober). Thus we have

Proposition 2.6.5. *X is sober iff the function $\beta_{\mathcal{O}X} \cdot \eta_X$ is a bijection.*

Combining Propositions 2.6.4 and 2.6.5, we obtain

Corollary 2.6.6. *X is sober iff*

$$(\forall \mathcal{F} \in \mathrm{cp}(\mathcal{O}X))(\exists! x \in X) \quad \mathcal{F} = \mathcal{U}_x.$$

Proposition 2.6.7. *X is T_0 iff $\eta_X : X \longrightarrow \mathrm{pt}(\mathcal{O}X)$ is injective.*

Proof. ($\Rightarrow$) Let X be T_0. Let $x, y \in X$ with $x \neq y$. Consider the frame maps $x^* : \mathcal{O}X \longrightarrow 2$ and $y^* : \mathcal{O}X \longrightarrow 2$. Then there exists $U \in \mathcal{O}X$ with $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$. Without loss of generality, suppose that it is $x \in U$ and $y \notin U$. Then $x^*(U) = 1$ and $y^*(U) = 0$. Hence $x^* \neq y^*$. That is, η_X is injective.

($\Leftarrow$) Let η_X be injective. Let $x, y \in X$ with $x \neq y$. Then $x^* \neq y^*$. So there exists $U \in \mathcal{O}X$ for which $x^*(U) \neq y^*(U)$. We either have $x^*(U) = 1$ or $x^*(U) = 0$. If $x^*(U) = 1$, then $x \in U$ and $y \notin U$. If $x^*(U) = 0$, then $x \notin U$ and $y \in U$. We can conclude that X is T_0. ■

Corollary 2.6.8. *Every sober space is T_0.*

Notice that Corollary 2.6.3 characterises sobriety in terms of *unique existential quantification*. We are thus motivated to weaken the notion of sobriety by omitting the 'uniqueness':

Definition 2.6.9. A topological space X is said to be *quasisober* iff

$$(\forall P \in \mathrm{pr}(\mathcal{O}X))(\exists x \in X) \quad P = X \setminus \overline{\{x\}}.$$

Proposition 2.6.10. *X is sober iff X is T_0 and quasisober.*

Proof. ($\Rightarrow$) This follows from Corollaries 2.6.3 and 2.6.8.

($\Leftarrow$) Let X be T_0 and quasisober. Since X is T_0, the map $\eta_X : X \longrightarrow \mathrm{pt}(OX)$ is injective, and hence the composite $\alpha_{OX} \cdot \eta_X : X \longrightarrow \mathrm{pr}(OX)$ is injective. Since X is quasisober, we have that, for any $P \in \mathrm{pr}(OX)$, there exists some $x \in X$ for which $P = X \setminus \overline{\{x\}}$ $(= \alpha_{OX}(\eta_X(x)))$, i.e. the composite $\alpha_{OX} \cdot \eta_X$ is surjective. It follows that $\alpha_{OX} \cdot \eta_X$ is bijective, and we can conclude that X is sober. ■

We can summarise the various characterisations of sobriety within the following theorem:

Theorem 2.6.11. *Let X be a topological space. The following statements are equivalent*:

(i) *The continuous map $\eta_X : X \longrightarrow \mathrm{pt}(OX)$ is a homeomorphism.*

(ii) *The continuous map $\eta_X : X \longrightarrow \mathrm{pt}(OX)$ is bijective.*

(iii) *There exists a locale A for which $X \cong \mathrm{pt}(A)$.*

(iv) *For any $P \in \mathrm{pr}(OX)$, there exists a unique $x \in X$ such that $P = X \setminus \overline{\{x\}}$.*

(v) *For any $\mathcal{F} \in \mathrm{cp}(OX)$, there exits a unique $x \in X$ such that $\mathcal{F} = \mathcal{U}_x$.*

(vi) *X is T_0 and, for any $P \in \mathrm{pr}(OX)$, there exists some $x \in X$ for which $P = X \setminus \overline{\{x\}}$.*

Proposition 2.6.12. *Every Hausdorff space is sober.*

Proof. Let X be a Hausdorff space. The empty space vacuously satisfies the sobriety axiom, and the one-point space trivially satisfies the sobriety axiom. If X only has two points, then the only topology on X which satisfies the Hausdorff axiom is the discrete topology, making OX a complete atomic Boolean algebra, and hence the primes of OX are precisely the complements of the atoms (the atoms are precisely the singletons – X is in particular T_1, so the singletons are closed), for which it follows that X is sober. So suppose that X has at least three points, and assume that X is not sober. Since X is T_0, it must be that X is not quasisober. Then there exists $P \in \mathrm{pr}(OX)$ such that, for each $x \in X$, $P \neq X \setminus \{x\}$ (X is T_1, so every singleton is closed). Fix such a prime P. Then (since also $P \neq X$) there exists $x, y \in X$ with $x \neq y$ such that $x, y \notin P$. Fix such x, y. Since X is Hausdorff, there exists $U, V \in OX$ with $U \cap V = \varnothing$ such that $x \in U$ and $y \in V$. Then $P \subsetneq P \cup U$ and $P \subsetneq P \cup V$. Now, it is straightforward to verify that $(P \cup U) \cap (P \cup V) = P$, and hence $P = P \cup U$ or $P = P \cup V$ – a contradiction. Therefore it must be the case that X is sober. ■

Remark 2.6.13. There is no implication in either direction between sobriety and the T_1 separation axiom; for examples illustrating this, see [13, 17, 33]. (The Sierpiński space is sober and not T_1, while any infinite set with cofinite topology is T_1 and not quasisober).

Chapter 3

Sublocales

In point-set topology, it is well-known that the subspace embeddings correspond to extremal monomorphisms[1] in **Top** [4]. The goal of this chapter is to understand the behaviour of subobjects in **Loc** – these correspond to the notion of sublocale as introduced in [13]. We will eventually describe a sublocale as a particular subset (satisfying certain lattice-theoretic properties) of a locale/frame, these subsets turn out to be the fixsets of special closure operators (called nuclei) on the frame.

Since **Loc** is an abstract category (the morphisms in **Loc** are *not* 'structure-preserving functions'), and we would like to view **Loc** as a generalisation of **Top**, it would be advantageous to modify **Loc** into a concrete category **Loc**$*$ in such a way that **Loc** $\cong$ **Loc**$*$. Indeed this is possible since frame homomorphisms have uniquely defined right adjoints – the localic maps.

Definition 3.0.1. We define the category **Loc**$*$ as the category whose objects are frames and whose morphisms are localic maps (recall Definition 2.1.4).

Remark 3.0.2. **Loc**$*$ is isomorphic to **Loc** – in this way **Loc**$*$ is a concrete representation of **Loc**. **Loc**$*$ is not to be confused with the formal opposite of **Frm** (this is of course the category **Loc** of locales): $\mathbf{Frm}^{\mathrm{op}} \neq \mathbf{Loc}*$, however we do have $\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}} \cong \mathbf{Loc}*$.

We will discuss the process in which subspaces induce sublocales, it is then straightforward to see how closed (resp. open) sublocales correspond to closed (resp. open) subspaces. The theory of locales diverges from that of topological spaces in that every locale contains a least dense sublocale (this fact is known as Isbell's Density Theorem) – this makes it possible to construct non-spatial sublocales from spatial locales.

The references used for this chapter are [13, 17, 31, 32].

[1] In a general category C a monomorphism m is said to be *extremal* iff whenever m can be factored as $m = f \cdot g$ where g is a bimorphism (i.e. g is monic and epic) then it follows that g is an isomorphism. Recall that if a composite $f \cdot g$ is monic, then g is also monic, thus for a monomorphism m to be extremally monic it suffices to ask for that in any decomposition $m = f \cdot e$ where e is an epimorphism then it follows that e is an isomorphism. The dual notion of extremal monomorphism is extremal epimorphism, that is e is extremally epic in C iff e^{op} is extremally monic in C^{op}. To spell this out, an epimorphism e is extremally epic iff whenever e can be factored as $e = m \cdot f$ where m is a monomorphism then it follows that m is an isomorphism.

3.1 Some technical machinery

We will need some categorical results which are of a technical nature, and whose proofs will be omitted from this thesis (though those who wish to read the proofs can consult the cited references).

Definition 3.1.1. A category C is said to be *concrete* iff there exists a covariant forgetful functor $U : C \longrightarrow \mathbf{Set}$ which is faithful.

Definition 3.1.2. A concrete category C is said to be *equationally presentable* iff C may be described as the category of sets equipped with certain operations satisfying certain equations involving these operations as objects and functions that preserve these operations as morphisms. (See [13, 26] for further details).

Definition 3.1.3. A concrete category C is said to be *algebraic* iff C is equationally presentable and the forgetful functor $U : C \longrightarrow \mathbf{Set}$ is a right adjoint.

Example 3.1.4. Frm is algebraic – it is equationally presentable, and in the next chapter we will outline how to construct the left adjoint of the forgetful functor $U : \mathbf{Frm} \longrightarrow \mathbf{Set}$.

Proposition 3.1.5. *In an algebraic category, the regular epimorphisms are precisely the extremal epimorphisms.* (*See* [29] *for further details*).

The following theorem is presented by Johnstone in [13], it will prove to be useful in this project.

Theorem 3.1.6. *Let C be an algebraic category. Then*:

(i) *C has all small limits, and they are constructed exactly as in* **Set** (*i.e. the forgetful functor $C \longrightarrow$* **Set** *preserves them*).

(ii) *C has all small colimits.*

(iii) *The monomorphisms in C are precisely the injective homomorphisms.*

(iv) *Epimorphisms in C need not be surjective, but the extremal epimorphisms in C are precisely the surjective homomorphisms.*

(v) *Every morphism in C can be factored* (*uniquely up to isomorphism*) *as an extremal epimorphism followed by a monomorphism.*

3.2 Closure operators

It will turn out that we will be able to represent sublocales as certain closure operators, we therefore provide the definition of closure operator that is used in this project.

Definition 3.2.1. A *closure operator* on a poset X is a monotone map $k : X \longrightarrow X$ such that, for each $x \in X$,

(i) $x \leqslant k(x)$ (k is *inflationary*), and

(ii) $kk(x) = k(x)$ (k is *idempotent*).

Notice that we can replace condition (ii) above by the condition

(ii)′ $kk(x) \leqslant k(x)$

since condition (i) guarantees $k(x) \leqslant kk(x)$. For a closure operator k on a poset X, we say an element $x \in X$ is k-*closed* iff $x = k(x)$ (equivalently, $k(x) \leqslant x$). The set $X_k := \{x \in X \mid x = k(x)\}$ is called the *fixset* of k in X. Notice that the fixset X_k is contained in the set $k[X]$. Now let $x \in X$, and consider $k(x)$. We have $k(x) = kk(x)$, therefore $k(x) \in X_k$, i.e. $k[X] \subseteq X_k$. We can conclude that $X_k = k[X]$. The map $k : X \longrightarrow X$ restricts to a surjective monotone map $k^* : X \twoheadrightarrow X_k$ where $k^*(x) = k(x)$.

Let $\begin{tikzcd} X \arrow[r, bend left, "f^*"] \arrow[r, phantom, "\perp"] & Y \arrow[l, bend left, "f_*"] \end{tikzcd}$ be a Galois connection. We know that $\mathrm{id}_X \leqslant f_* f^*$ and $f^* f_* f^* = f^*$, from which it follows that the monotone map $f_* f^* : X \longrightarrow X$ is inflationary and idempotent, i.e. $f_* f^*$ is a closure operator on X.

Let $k : X \longrightarrow X$ be a closure operator. Let $j : X_k \hookrightarrow X$ be the inclusion of the fixset X_k into X. We claim that $k^* : X \twoheadrightarrow X_k$ is left adjoint to j. Indeed, for $x \in X$ and $x' \in X_k$,

$$\begin{aligned} k(x) \leqslant x' &\implies x \leqslant x' \\ &\iff x \leqslant j(x'), \end{aligned}$$

and conversely,

$$\begin{aligned} x \leqslant j(x') &\iff x \leqslant x' \\ &\implies k(x) \leqslant k(x') \\ &\iff k(x) \leqslant x'. \end{aligned}$$

Thus $k^* \dashv j$, and we will use the notation $k_* := j$; moreover $k^* \cdot j = \mathrm{id}_{X_k}$ (since k^* is surjective). We bear in mind the slogan "closure operators are left adjoints".

3.3 Introducing sublocales

In **Frm** every frame homomorphism $h : A \longrightarrow B$ factors as $h = j \cdot \overline{h}$ where $\overline{h} : A \twoheadrightarrow h[A]$ is the surjective frame map (extremal epimorphism) $a \longmapsto h(a)$, and $j : h[A] \hookrightarrow B$ is the inclusion (monomorphism) of $h[A]$ into B. This is of course one of the consequences of **Frm** being algebraic (recall Theorem 3.1.6).

Remark 3.3.1. Surjective frame maps will be referred to as *sublocale homomorphisms*.

Dually, in **Loc** every morphism $f : B \longrightarrow A$ factors as

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & A \\ & f^*[A] \arrow[ur, tail, "m"'] & \end{tikzcd}$$

where e is an epimorphism ($e^* = j : f^*[A] \hookrightarrow B$ is the inclusion), and m is an extremal monomorphism ($m^* = \overline{f^*} : A \longrightarrow f^*[A]$ is the surjection $a \longmapsto f^*(a)$). In short, every locale morphism factors as an epimorphism followed by an extremal monomorphism.

3.3.1 Isomorphism classes

Let C be a category, and fix an object A in C. Two morphisms $B \xrightarrow{f} A$ and $C \xrightarrow{g} A$ are said to be *isomorphic* (notated $f \cong g$) iff there exists an isomorphism $\alpha : B \longrightarrow C$ such that $f = g \cdot \alpha$. The relation $\cong$ on the class of morphisms in C with codomain A is clearly an equivalence relation; the equivalence class containing a morphism $f : B \longrightarrow A$ will be notated as $[f]$ and will be referred to as 'the isomorphism class containing f'.

Definition 3.3.2. A *sublocale* of a locale A is an isomorphism class of extremal monomorphisms in **Loc** with codomain A.

For a locale A, we write $\mathsf{Sub}(A)$ for the set of all sublocales of A.

3.4 Nuclei and sublocales

Let $f : B \longrightarrow A$ be a locale morphism. f induces a Galois connection

$$\begin{tikzcd} A \arrow[r, bend left, "f^*"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f_*"] \end{tikzcd}$$

where f^* is a frame homomorphism, and f_* is a localic map. The composite $f_* f^*$ is thus a closure operator on A; moreover, since f^* and f_* both preserve finite meets, $f_* f^*$ preserves finite meets. This motivates the following definition:

Definition 3.4.1. A *nucleus* on a frame A is a closure operator $n : A \longrightarrow A$ for which, for all $a, a' \in A$, $n(a \wedge a') = n(a) \wedge n(a')$.

For a nucleus n on a frame A we have $a \leqslant n(a)$ for all $a \in A$, so it follows that $n(1) = 1$, and hence n preserves all finite meets. The composite $f_* f^*$ is called the nucleus induced by the morphism f. Being consistent with notation, we write A_n for the fixset of n in A. Between posets, we have a surjective monotone map $n^* : A \twoheadrightarrow A_n$ (defined by $a \longmapsto n(a)$) whose right adjoint is the inclusion $n_* = j : A_n \hookrightarrow A$. We write $N(A)$ for the set of nuclei on A.

Lemma 3.4.2. *A_n (with the inherited order) is a frame, and $n^* : A \twoheadrightarrow A_n$ is a frame homomorphism.*

Proof.

1. A_n is closed under finite meets in A.

Firstly, note that $1 \in A_n$ (i.e. A_n is closed under the empty meet in A). Let $a, a' \in A_n$, we need to verify that $a \wedge a' \in A_n$. We have $a = n(a)$ and $a' = n(a')$, hence $a \wedge a' = n(a) \wedge n(a') = n(a \wedge a')$ – since n preserves all finite meets. Thus $a \wedge a' \in A_n$. It follows that A_n has finite meets, which are the same as those in A.

2. <u>A_n has arbitrary joins.</u>

Let $M \subseteq A_n$, and consider M's join in A: $\bigvee M$. We have $n(\bigvee M) \in A_n$ and, for each $m \in M$, $m \leqslant \bigvee M \leqslant n(\bigvee M)$. Therefore $n(\bigvee M)$ is an upper bound of M in A_n. Let $\alpha \in A_n$ be any upper bound of M in A_n. Then obviously α is an upper bound of M in A, hence $\bigvee M \leqslant \alpha$, so $n(\bigvee M) \leqslant n(\alpha) = \alpha$. This shows that $n(\bigvee M)$ is the least upper bound of M in A_n. Therefore A_n has arbitrary joins, and, for any subset $M \subseteq A_n$, we denote its join in A_n as $\bigsqcup M := n(\bigvee M)$. It follows that A_n is a complete lattice.

3. <u>$n^* : A \twoheadrightarrow A_n$ preserves finite meets and arbitrary joins.</u>

We already know that $n^*(1) = 1$. Take $a, a' \in A$. Then $n^*(a \wedge a') = n^*(a) \wedge n^*(a')$, since n is a nucleus and preserves binary meets. It follows that n^* preserves finite meets. Take a subset $S \subseteq A$. For each $s \in S$, $s \leqslant \bigvee S$, hence $n^*(s) \leqslant n^*(\bigvee S)$, which yields $\bigsqcup n^*[S] \leqslant n^*(\bigvee S)$ in A_n. Also, for each $s \in S$, $s \leqslant n^*(s)$, so $\bigvee S \leqslant \bigvee n^*[S]$ in A – thus $n^*(\bigvee S) \leqslant n^*(\bigvee n^*[S]) = \bigsqcup n^*[S]$ in A_n. It follows that n^* preserves arbitrary joins.

4. <u>A_n satisfies the frame distributivity law.</u>

Take $a \in A_n$ and $M \subseteq A_n$. We have

$$\begin{aligned} a \wedge \bigsqcup M &= a \wedge n(\bigvee M) \\ &= n(a) \wedge n(\bigvee M) \\ &= n(a \wedge \bigvee M) \\ &= n(\bigvee \{a \wedge m \mid m \in M\}) \\ &= \bigsqcup \{a \wedge m \mid m \in M\}. \end{aligned}$$

■

So for a nucleus $n : A \longrightarrow A$, we know that A_n is a frame and the map $n^* : A \twoheadrightarrow A_n$ is a surjective frame homomorphism whose right adjoint is the localic map $n_* = j : A_n \hookrightarrow A$.

Theorem 3.4.3. *For any locale A, there is a bijection between* $\mathsf{Sub}(A)$ *and* $N(A)$.

Proof. Fix a locale A. Let $[f]$ be a sublocale of A. Then we have an extremal monomorphism $f : B \longrightarrow A$ in **Loc** which yields a nucleus $f_* f^* : A \longrightarrow A$. If $\widetilde{f} : C \longrightarrow A$ is an extremal monomorphism in **Loc** that belongs to the isomorphism class $[f]$, then $f = \widetilde{f} \cdot \alpha$ for some isomorphism $\alpha : B \longrightarrow C$. We have $f^* = (\widetilde{f} \cdot \alpha)^* = \alpha^* \cdot \widetilde{f}^*$, since $(-)^*$ is a contravariant functor from **Loc** to **Frm**, also $f_* = (\widetilde{f} \cdot \alpha)_* = \widetilde{f}_* \cdot \alpha_*$, since $(-)_*$ is a covariant functor from **Loc** to **Loc**∗. So $f_* f^* = \widetilde{f}_* \alpha_* \alpha^* \widetilde{f}^* = \widetilde{f}_* \widetilde{f}^*$, since $\alpha_* \alpha^* = \mathrm{id}_C$ because $\alpha^* \dashv \alpha_*$ with α_* surjective. It follows that we have a well-defined function $\mathsf{Sub}(A) \longrightarrow N(A)$. Now, let $n : A \longrightarrow A$ be a nucleus. Let $g : A_n \rightarrowtail A$ be the locale morphism corresponding to the surjective frame homomorphism $n^* : A \twoheadrightarrow A_n$, i.e. $g^* = n^*$. Note that g is an extremal monomorphism in **Loc**. We also have $g_* = n_* = j : A_n \hookrightarrow A$ (an injective localic map). Consider the commutative diagram (in **Pos** = the category of posets and monotone maps)

$$\begin{tikzcd} A \arrow[r, two heads, "g^*"] \arrow[rr, bend right, "n"'] & A_n \arrow[r, hook, "g_*"] & A. \end{tikzcd}$$

Let $m : B \rightarrowtail A$ be an extremal monomorphism in **Loc** such that $n = m_* m^*$, i.e. so that the diagram (in **Pos**)

$$\begin{tikzcd} A \arrow[r, two heads, "m^*"] \arrow[rr, bend right, "n"'] & B \arrow[r, tail, "m_*"] & A \end{tikzcd}$$

commutes. We wish to show that $g \cong m$, i.e. that $g^* \cong m^*$. We must exhibit an isomorphism of frames $\varphi : B \longrightarrow A_n$ such that $g^* = \varphi \cdot m^*$. Consider the commutative diagram (in **Pos**)

$$\begin{tikzcd} A \arrow[r, two heads, "g^*"] \arrow[dr, "m^*"'] & A_n \arrow[r, hook, "g_*"] & A \\ & B. \arrow[ur, tail, "m_*"'] & \end{tikzcd}$$

We have $A_n = g^*[A] = g_*g^*[A] = m_*m^*[A] = m_*[B]$. Let $\varphi : B \longrightarrow A_n$ be the map $b \longmapsto m_*(b)$. It follows that φ is a monotone bijection of posets. We claim that φ is order-reflecting. Suppose $a \leqslant a'$ in A_n. Then $a = \varphi(b) \leqslant \varphi(b') = a'$ for some $b, b' \in B$, so $m_*(b) \leqslant m_*(b')$. Then $m^*m_*(b) \leqslant m^*m_*(b')$, but $m^*m_* = \mathrm{id}_B$ because $m^* \dashv m_*$ where m^* is surjective and m_* is injective. Therefore $b \leqslant b'$. It follows that φ is an order-embedding of B onto A_n, that is, φ is an order-isomorphism between frames and hence an isomorphism in **Frm**. Now, for each $a \in A$, $\varphi(m^*(a)) = m_*(m^*(a)) = g_*(g^*(a)) = g^*(a)$. Therefore $g^* = \varphi \cdot m^*$. ∎

In view of Theorem 3.4.3 and Lemma 3.4.2, we now modify the definition of sublocale:

Definition 3.4.4. Let A be a locale. A *sublocale* of A is a fixset A_n for some nucleus $n : A \longrightarrow A$.

Going forward, whenever we use the term ‘sublocale’, we will mean the notion as defined in Definition 3.4.4.

Corollary 3.4.5. *A subset S of a locale A is a sublocale iff S is a locale (with respect to the inherited order) and the inclusion $j : S \hookrightarrow A$ is a localic map.*

Proof. ($\Rightarrow$) Let S be a sublocale. Then $S = A_n$ for some nucleus $n : A \longrightarrow A$. We know that $n^* : A \twoheadrightarrow S$ is a frame homomorphism, and hence $n_* = j : S \hookrightarrow A$ is a localic map.

($\Leftarrow$) Suppose S is a locale (with respect to the inherited order) and $j : S \hookrightarrow A$ is localic. Then $j^* : A \twoheadrightarrow S$ is a frame homomorphism (where $j^* \dashv j$). The composite $n := j \cdot j^*$ is a nucleus on A and $S = A_n$. ∎

There is a very useful lattice-theoretic characterisation of sublocales:

Theorem 3.4.6. *A subset S of a locale A is a sublocale iff:*

(i) S is closed under arbitrary meets in A, and

(ii) for all $a \in A$ and $s \in S$, $(a \Rightarrow s) \in S$.

Proof. ($\Rightarrow$) Let $S \subseteq A$ be a sublocale. Then there is a nucleus $n : A \longrightarrow A$ for which $S = A_n$. Of course, A_n is a frame, and the inclusion $j : A_n \hookrightarrow A$ is a localic map. Let $M \subseteq A_n$, and write $\sqcap M$ for the meet of M in A_n. Since j preserves all meets, $\sqcap M = j(\sqcap M) = \bigwedge j[M] = \bigwedge M$. Therefore $\bigwedge M \in A_n$, and it follows that the meets in A_n are the same as those in A. Take $a \in A$ and $a' \in A_n$. We have $(a \Rightarrow a') = \max\{x \in A \mid a \wedge x \leqslant a'\}$. Set $y = (a \Rightarrow a')$. We need to show that $y \in A_n$, i.e. we must show that $y = n(y)$. We have $a \wedge y \leqslant a'$, hence $n(a \wedge y) = n(a) \wedge n(y) \leqslant n(a') = a'$. Now, $a \wedge n(y) \leqslant n(a) \wedge n(y) \leqslant a'$. So $n(y) \in \{x \in A \mid a \wedge x \leqslant a'\}$, which yields $n(y) \leqslant y$. We can conclude that $y = n(y)$.

($\Leftarrow$) Let $S \subseteq A$ be such that S is closed under arbitrary meets in A, and $(\forall a \in A)(\forall s \in S)\ (a \Rightarrow s) \in S$. We must exhibit a nucleus $n : A \longrightarrow A$ for which $S = A_n$ (i.e. $S = n[A]$). Define a function $n : A \longrightarrow A$ by $n(a) = \bigwedge\{x \in S \mid a \leqslant x\}$. Since S is closed under meets in A, it follows that, for each $a \in A$, $n(a) \in S$, i.e. $n[A] \subseteq S$. Let $s \in S$, then trivially $s = \bigwedge\{x \in S \mid s \leqslant x\}$, so $s \in n[A]$. Therefore $S = n[A]$. We claim that n is a nucleus on A. Firstly, since S is closed under meets in A, it follows that $1 \in S$ and $n(1) = \bigwedge\{x \in S \mid 1 \leqslant x\} = 1$ – so n preserves the empty meet. Let $a \in A$, and let $\alpha \in S$ be such that $\alpha = n(a)$ $(= \bigwedge\{x \in S \mid a \leqslant x\})$. If $a \in S$, then $\alpha = a$, otherwise $a < \alpha$. So $a \leqslant \alpha$, that is, $n : A \longrightarrow A$ is inflationary. Let $a, a' \in A$. We have $n(a \wedge a') = \min\{x \in S \mid a \wedge a' \leqslant x\}$. Since n is inflationary, $a \wedge a' \leqslant n(a \wedge a')$ and $a \wedge a' \leqslant n(a) \wedge n(a')$. So $n(a) \wedge n(a') \in \{x \in S \mid a \wedge a' \leqslant x\}$, which yields $n(a \wedge a') \leqslant n(a) \wedge n(a')$. Now consider

$$\begin{aligned}
a \wedge a' \leqslant n(a \wedge a') \quad &\text{iff} \quad a' \leqslant (a \Rightarrow n(a \wedge a')) \\
&\text{iff} \quad n(a') \leqslant (a \Rightarrow n(a \wedge a')) \text{ because } n(a') = \min\{x \in S \mid a' \leqslant x\} \\
&\text{iff} \quad a \wedge n(a') \leqslant n(a \wedge a') \\
&\text{iff} \quad a \leqslant (n(a') \Rightarrow n(a \wedge a')) \\
&\text{iff} \quad n(a) \leqslant (n(a') \Rightarrow n(a \wedge a')) \\
&\text{iff} \quad n(a) \wedge n(a') \leqslant n(a \wedge a').
\end{aligned}$$

We now know that n preserves all finite meets (and hence is monotone). For $a \in A$, $nn(a) = \bigwedge\{x \in S \mid n(a) \leqslant x\}$, but $n(a) \in S$, so $nn(a) = n(a)$ (n is idempotent). We can conclude that $n : A \longrightarrow A$ is a nucleus and $S = A_n$. ∎

Proposition 3.4.7. *If $f : A \longrightarrow B$ is a localic map, and $S \subseteq A$ is a sublocale, then $f[S]$ is a sublocale of B; moreover, the restriction $f_{|S} : S \twoheadrightarrow f[S]$ of f from S onto $f[S]$ is a localic map.*

Proof. Let $f : A \longrightarrow B$ be a localic map, and let $S \subseteq A$ be a sublocale. Let $M \subseteq f[S]$. Then $M = f[E]$ for some $E \subseteq S$. We have $\bigwedge M = \bigwedge f[E] = f(\bigwedge E) \in f[S]$ because S is closed under meets in A. Take $b \in B$ and $x \in f[S]$ (i.e. $x = f(s)$ for some $s \in S$). We need to show that $(b \Rightarrow f(s)) \in f[S]$. Since S is a sublocale, we have $(f^*(b) \Rightarrow s) \in S$, hence $f(f^*(b) \Rightarrow s) \in f[S]$. But, by the Frobenius identity for localic maps (recall Corollary 2.1.8 in the previous chapter) $f(f^*(b) \Rightarrow s) = (b \Rightarrow f(s))$. It follows that $f[S]$ is a sublocale of B. Let $f_{|S} : S \twoheadrightarrow f[S]$ be the map $S \ni s \longmapsto f(s)$. For $M \subseteq S$, $f_{|S}(\bigwedge M) = f(\bigwedge M) = \bigwedge f[M] = \bigwedge f_{|S}[M] \in f[S]$ – so $f_{|S}$ preserves all meets and hence has a left adjoint $f^*_{|S} : f[S] \rightarrowtail S$. We need to verify that $f^*_{|S}$ preserves all finite meets. We have

$$\begin{aligned}
(\forall x \in f[S])(\forall s \in S) \quad f^*_{|S}(x) \leqslant s &\text{ iff } x \leqslant f_{|S}(s) \\
&\text{ iff } x \leqslant f(s) \\
&\text{ iff } f^*(x) \leqslant s,
\end{aligned}$$

hence, for each $x \in f[S]$, $f^*_{|S}(x) = f^*(x)$. Since f^* preserves all finite meets, it follows that $f^*_{|S}$ preserves all finite meets. We can conclude that $f_{|S}$ is a localic map. ∎

Observation 3.4.8. We can conclude that, in the concrete category **Loc**$*$, any localic map $f : A \longrightarrow B$ factors as

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "f_{|A}"'] & & B \\
& f[A]. \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

In particular, if $f : A \longrightarrow B$ is an injective localic map, then A is isomorphic to a sublocale of B, namely $f[A]$.

3.5 Sublocales induced by subspaces

As a consequence of Proposition 3.4.7 and Observation 3.4.8, we can construct sublocales of spatial frames which are induced by subspaces. We will now spell this out. Let X be a topological space and $Y \subseteq X$ a subspace. Let $j : Y \hookrightarrow X$ be the inclusion of Y into X. We obtain a surjective frame homomorphism $j^* : \mathcal{O}X \twoheadrightarrow \mathcal{O}Y$, where, for each $V \in \mathcal{O}X$, $j^*(V) = j^{-1}[V] = Y \cap V$. This yields an injective localic map $j_* : \mathcal{O}Y \rightarrowtail \mathcal{O}X$. Thus $j_*[\mathcal{O}Y]$ is a sublocale of $\mathcal{O}X$ and $\mathcal{O}Y \cong j_*[\mathcal{O}Y]$. We write $S(Y) := j_*[\mathcal{O}Y]$, and call $S(Y)$ *the sublocale induced by the subspace Y*.

Observation 3.5.1. Each sublocale of a spatial frame $\mathcal{O}X$ that is induced by a subspace of X is itself a spatial frame.

Proposition 3.5.2. *For each $U \in \mathcal{O}Y$, $j_*(U) = \mathrm{int}_X((X \setminus Y) \cup U)$.*

Proof. By Proposition 2.1.6 in the previous chapter, $j_*(U) = X \setminus \overline{j[Y \setminus U]} = X \setminus \overline{(Y \setminus U)}$. Now

$$\begin{aligned} Y \setminus U &= Y \cap (X \setminus U) \\ &= (X \setminus (X \setminus Y)) \cap (X \setminus U) \\ &= X \setminus ((X \setminus Y) \cup U). \end{aligned}$$

Thus

$$\begin{aligned} j_*(U) &= X \setminus \overline{(X \setminus ((X \setminus Y) \cup U))} \\ &= \mathrm{int}_X((X \setminus Y) \cup U). \end{aligned}$$

■

Observation 3.5.3. $j_*(U)$ can be interpreted as the largest open set in X whose intersection with Y yields U.

Proposition 3.5.4. *For any topological space X and subspace $Y \subseteq X$, the sublocale of $\mathcal{O}X$ that is induced by Y is determined by the formula*

$$S(Y) = \{\mathrm{int}_X((X \setminus Y) \cup V) \mid V \in \mathcal{O}X\}.$$

Proof. We have

$$\begin{aligned} S(Y) &= j_*[\mathcal{O}Y] \\ &= \{\mathrm{int}_X((X \setminus Y) \cup U) \mid U \in \mathcal{O}Y\} \\ &= \{\mathrm{int}_X((X \setminus Y) \cup (Y \cap V)) \mid V \in \mathcal{O}X\} \\ &= \{\mathrm{int}_X(((X \setminus Y) \cup Y) \cap ((X \setminus Y) \cup V)) \mid V \in \mathcal{O}X\} \\ &= \{\mathrm{int}_X((X \setminus Y) \cup V) \mid V \in \mathcal{O}X\}. \end{aligned} \tag{3.1}$$

■

3.6 Closed sublocales

Let X be a topological space, and let Y be a closed subspace of X. Then, for each $V \in \mathcal{O}X$, $\mathrm{int}_X((X \setminus Y) \cup V) = (X \setminus Y) \cup V$, thus Eq. (3.1) reads

$$\begin{aligned} S(Y) &= \{(X \setminus Y) \cup V \mid V \in \mathcal{O}X\} \\ &= \{V \in \mathcal{O}X \mid (X \setminus Y) \subseteq V\} \\ &= \uparrow(X \setminus Y). \end{aligned}$$

It is straightforward to verify that in an arbitrary locale A, for $a \in A$, the up-set $\uparrow a$ is a sublocale. We are therefore motivated to say

Definition 3.6.1. For any locale A, and $a \in A$, the sublocales of the form $\mathfrak{c}(a) := \uparrow a$ are said to be *closed*.

What is the nucleus corresponding to the closed sublocale $\uparrow a$? Let us denote this nucleus by $c(a)$. For $x \in A$, we have

$$\begin{aligned} c(a)(x) &= \bigwedge\{y \in \uparrow a \mid x \leqslant y\} \qquad \text{(recall the proof of Theorem 3.4.6)} \\ &= \bigwedge\{y \in A \mid a \leqslant y \text{ and } x \leqslant y\} \\ &= a \vee x. \end{aligned}$$

Thus the *closed nucleus* corresponding to the element $a \in A$ is given by $c(a) := a \vee (-)$.

Proposition 3.6.2. *A sublocale S of a spatial frame $\mathcal{O}X$ is closed iff S is induced by a closed subspace of X.*

Proof. ($\Rightarrow$) Let S be a closed sublocale of $\mathcal{O}X$. Then $S = \uparrow V$ for some $V \in \mathcal{O}X$. Clearly $\uparrow V$ is induced by the closed subspace $X \setminus V$: $S(X \setminus V) = \uparrow(X \setminus (X \setminus V)) = \uparrow V$.

($\Leftarrow$) If S is induced by a closed subspace of X, then $S = S(Y)$ where $Y \subseteq X$ is some closed subspace, and $S(Y) = \uparrow(X \setminus Y)$ which is a closed sublocale of $\mathcal{O}X$. ■

Corollary 3.6.3. *Closed sublocales of spatial frames are always spatial.*

For any locale A, we write $\mathsf{Closed}(A)$ for the poset of closed sublocales of A ordered by inclusion. We have a map $\mathfrak{c} : A \longrightarrow \mathsf{Closed}(A)$ where $\mathfrak{c}(a) := \uparrow a$. $\mathfrak{c}$ is clearly a bijection; moreover, $a \leqslant b$ in A iff $\uparrow b \subseteq \uparrow a$ in $\mathsf{Closed}(A)$. Therefore we have established

Theorem 3.6.4. *For any locale A, A^{op} is isomorphic to* $\mathsf{Closed}(A)$.

3.7 Open sublocales

Lemma 3.7.1. *For any topological space X, the Heyting implication on $\mathcal{O}X$ is given by the formula*

$$U \Rightarrow V = \mathrm{int}_X((X \setminus U) \cup V)$$

where int_X is the interior operator on X.

Proof. Let $U, V \in \mathcal{O}X$. Then $U \Rightarrow V = \bigcup\{W \in \mathcal{O}X \mid U \cap W \subseteq V\}$. We have

$$\begin{aligned} U \cap W \subseteq V &\iff (X \setminus U) \cup (U \cap W) \subseteq (X \setminus U) \cup V \\ &\iff (X \setminus U) \cup W \subseteq (X \setminus U) \cup V \\ &\iff W \subseteq (X \setminus U) \cup V. \end{aligned}$$

Thus $U \Rightarrow V = \bigcup\{W \in \mathcal{O}X \mid W \subseteq (X \setminus U) \cup V\} = \mathrm{int}_X((X \setminus U) \cup V)$. ∎

Let Y be an open subspace of a topological space X. Then

$$\begin{aligned} S(Y) &= \{\mathrm{int}_X((X \setminus Y) \cup V) \mid V \in \mathcal{O}X\} \\ &= \{Y \Rightarrow V \mid V \in \mathcal{O}X\}. \end{aligned}$$

Again, it's straightforward to check that in any locale A, with $a \in A$, the subset $\{a \Rightarrow x \mid x \in A\}$ is a sublocale of A. We thus have

Definition 3.7.2. For any locale A, and $a \in A$, the sublocales of the form $\mathfrak{o}(a) := \{a \Rightarrow x \mid x \in A\}$ are said to be *open*.

For $a \in A$, $\mathfrak{o}(a)$ is obviously the image of A under the map $o(a) := a \Rightarrow (-)$; it can easily be verified that $o(a)$ is a nucleus (see [13]): it is the *open nucleus* corresponding to a.

Now fix a locale A, and take $a \in A$. Then $\downarrow a$ is itself a locale, and we can consider the locale morphism $f : \downarrow a \longrightarrow A$ defined by the frame homomorphism $f^* : A \longrightarrow \downarrow a$ where $f^*(x) = a \wedge x$. f^* is clearly surjective, and thus f is an extremal monomorphism in **Loc**. If we consider the injective localic map $f_* : \downarrow a \rightarrowtail A$, then we know that $\downarrow a$ is isomorphic to the sublocale $f_*[\downarrow a]$. Let $y \in \downarrow a$. Then $f_*(y) = \bigvee\{x \in A \mid a \wedge x \leqslant y\} = (a \Rightarrow y)$. Thus $\downarrow a \cong \{a \Rightarrow y \mid y \in A \text{ and } y \leqslant a\}$. But $\{a \Rightarrow y \mid y \in A \text{ and } y \leqslant a\} = \{a \Rightarrow (a \wedge x) \mid x \in A\} = \{(a \Rightarrow a) \wedge (a \Rightarrow x) \mid x \in A\} = \{a \Rightarrow x \mid x \in A\} = \mathfrak{o}(a)$. Therefore $f_*[\downarrow a] = \mathfrak{o}(a)$, and we can conclude that $\downarrow a \cong \mathfrak{o}(a)$.

Proposition 3.7.3. *A sublocale S of a spatial frame $\mathcal{O}X$ is open iff S is induced by an open subspace of X.*

Proof. ($\Rightarrow$) Let S be an open sublocale of $\mathcal{O}X$. Then $S = \{U \Rightarrow V \mid V \in \mathcal{O}X\}$ for some $U \in \mathcal{O}X$. Clearly S is induced by the open subspace U.

($\Leftarrow$) Let S be induced by an open subspace U of X. Then $S = S(U) = \{U \Rightarrow V \mid V \in \mathcal{O}X\}$ – which is an open sublocale of $\mathcal{O}X$. ∎

Corollary 3.7.4. *Open sublocales of spatial frames are always spatial.*

Moreover, in [17], Johnstone shows that any locale A is isomorphic to its lattice of open sublocales $\mathsf{Open}(A)$ – also see [39].

3.8 A detour on negation in Heyting algebras

Recall that in a Boolean algebra, we have $a \Rightarrow b = \neg a \vee b$ (where $\neg a$ is the complement of a), and hence $\neg a = a \Rightarrow 0$. This leads us to

Definition 3.8.1. For an element a in a Heyting algebra A, we define $\neg a := a \Rightarrow 0$, and say that $\neg a$ is the *negation* of a. $\neg a$ is the largest element $x \in A$ for which $a \wedge x = 0$.

Remark 3.8.2. Many authors refer to $\neg a$ as the *pseudocomplement* of a, and write $a^* = \neg a$.

Observation 3.8.3. In any Heyting algebra, $\neg 0 = 1$ and $\neg 1 = 0$; moreover, $a \leqslant \neg b$ iff $b \wedge a = 0$.

Proposition 3.8.4. *Let A be a Heyting algebra. The negation operator $\neg : A \longrightarrow A$ is order-reversing.*

Proof. Let $a \leqslant b$ in A. Then $a \wedge \neg b \leqslant b \wedge \neg b = 0$, while $\neg a$ is the greatest element x in A with the property $a \wedge x = 0$, so $\neg b \leqslant \neg a$. ■

Corollary 3.8.5. *For any Heyting algebra A, the composite $\neg\neg : A \longrightarrow A$ is monotone and inflationary.*

Proof. That $\neg\neg : A \longrightarrow A$ is monotone follows immediately from Proposition 3.8.4. For $a \in A$, we have $\neg a \wedge a = 0$, hence $a \leqslant \neg\neg a$. ■

Proposition 3.8.6. *For any element a in a Heyting algebra A, we have $\neg\neg\neg a = \neg a$.*

Proof. We have $\neg a \leqslant \neg\neg(\neg a)$, because $\neg\neg$ is inflationary. Since $a \leqslant \neg\neg a$, we have $\neg\neg\neg a \leqslant \neg a$ (because $\neg$ is order-reversing). ■

Proposition 3.8.7. *For any a and b in a Heyting algebra A, we have*

$$\neg(a \vee b) = \neg a \wedge \neg b.$$

Proof. We have $a \leqslant a \vee b$, so $\neg(a \vee b) \leqslant \neg a$. Similarly, $\neg(a \vee b) \leqslant \neg b$. Hence $\neg(a \vee b) \leqslant \neg a \wedge \neg b$. Now,

$$\begin{aligned}
(a \vee b) \wedge (\neg a \wedge \neg b) &= (a \wedge (\neg a \wedge \neg b)) \vee (b \wedge (\neg a \wedge \neg b)) \\
&= ((a \wedge \neg a) \wedge \neg b) \vee ((b \wedge \neg b) \wedge \neg a) \\
&= (0 \wedge \neg b) \vee (0 \wedge \neg a) \\
&= 0.
\end{aligned}$$

Thus, $\neg a \wedge \neg b \leqslant \neg(a \vee b)$. ■

Proposition 3.8.8. *For any a and b in a Heyting algebra A, we have*

$$\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b.$$

Proof. We have $a \wedge b \leqslant a$, so $\neg\neg(a \wedge b) \leqslant \neg\neg a$. Similarly, $\neg\neg(a \wedge b) \leqslant \neg\neg b$. Hence $\neg\neg(a \wedge b) \leqslant \neg\neg a \wedge \neg\neg b$. Now,

$$\begin{aligned}
& \neg\neg a \wedge \neg\neg b \leqslant \neg\neg(a \wedge b) \\
\text{iff}\quad & \neg\neg a \wedge \neg\neg b \wedge \neg(a \wedge b) = 0 \\
\text{iff}\quad & \neg\neg b \wedge \neg(a \wedge b) \leqslant \neg\neg\neg a \\
\text{iff}\quad & \neg\neg b \wedge \neg(a \wedge b) \leqslant \neg a \\
\text{iff}\quad & \neg\neg b \wedge \neg(a \wedge b) \wedge a = 0 \\
\text{iff}\quad & \neg(a \wedge b) \wedge a \leqslant \neg\neg\neg b \\
\text{iff}\quad & \neg(a \wedge b) \wedge a \leqslant \neg b \\
\text{iff}\quad & \neg(a \wedge b) \wedge (a \wedge b) = 0.
\end{aligned}$$

And since we always have $\neg(a \wedge b) \wedge (a \wedge b) = 0$, it follows that $\neg\neg a \wedge \neg\neg b \leqslant \neg\neg(a \wedge b)$. ■

Definition 3.8.9. For a Heyting algebra A, we say that *excluded middle holds* iff, for each $a \in A$, $a \vee \neg a = 1$.

Trivially, we see that

Observation 3.8.10. A Heyting algebra is a Boolean algebra iff excluded middle holds.

Definition 3.8.11. In any Heyting algebra A, an element $a \in A$ is said to be *regular* iff $a = \neg\neg a$. We write $A_{\neg\neg}$ for the set of regular elements in A, and we give $A_{\neg\neg}$ the inherited order.

Proposition 3.8.12. *A Heyting algebra A is a Boolean algebra iff every element of A is regular.*

Proof. ($\Rightarrow$) Let A be a Boolean algebra, and take $a \in A$. We have $\neg a \vee \neg\neg a = 1$ and $\neg a \wedge \neg\neg a = 0$, that is, $\neg\neg a$ is the *unique* complement of $\neg a$: $\neg\neg a = a$.

($\Leftarrow$) Suppose that every element of A is regular, and consider $a \in A$. We have

$$\begin{aligned} a \vee \neg a &= \neg\neg(a \vee \neg a) \\ &= \neg(\neg a \wedge \neg\neg a) \\ &= \neg(\neg a \wedge a) \\ &= \neg 0 \\ &= 1. \end{aligned}$$

Thus, A is a Boolean algebra. ∎

Proposition 3.8.13. *For any Heyting algebra A, $A_{\neg\neg}$ is a Boolean algebra.*

Proof. Let us first show that $A_{\neg\neg}$ is a lattice. Since $\neg 0 = 1$ and $\neg 1 = 0$, it is clear that $0, 1 \in A_{\neg\neg}$. Take $a, b \in A_{\neg\neg}$. Then $a \wedge b = \neg\neg a \wedge \neg\neg b = \neg\neg(a \wedge b)$. Thus $a \wedge b \in A_{\neg\neg}$; it follows that $A_{\neg\neg}$ is a sub-meet-semilattice of A. What is binary join in $A_{\neg\neg}$? Note that, for all $x \in A$, $\neg\neg x = \neg\neg\neg\neg x$ (because $\neg x = \neg\neg\neg x$), so $\neg\neg x \in A_{\neg\neg}$. Thus, for $a, b \in A_{\neg\neg}$, $\neg\neg(a \vee_A b) \in A_{\neg\neg}$. Since $a, b \leqslant a \vee_A b \leqslant \neg\neg(a \vee_A b)$, it is clear that $\neg\neg(a \vee_A b)$ is an upper bound of $\{a, b\}$ in $A_{\neg\neg}$. Let $a' \in A_{\neg\neg}$ be any upper bound of $\{a, b\}$ in $A_{\neg\neg}$. Then clearly $a \vee_A b \leqslant a'$, hence $\neg\neg(a \vee_A b) \leqslant \neg\neg a' = a'$, and so it follows that $\neg\neg(a \vee_A b)$ is the least upper bound of $\{a, b\}$ in $A_{\neg\neg}$. Next we claim that $A_{\neg\neg}$ is a Heyting algebra. For $a, b \in A_{\neg\neg}$, put $a \Rightarrow_{\neg\neg} b = a \Rightarrow b$ (where $\Rightarrow$ is the Heyting implication on A). Let $c \in A_{\neg\neg}$. Then $a \wedge c \leqslant b$ iff $c \leqslant a \Rightarrow b$ iff $c \leqslant a \Rightarrow_{\neg\neg} b$. Thus $A_{\neg\neg}$ is a Heyting algebra, and the Heyting implication in $A_{\neg\neg}$ coincides with that in A, which further implies that the negation operator in $A_{\neg\neg}$ coincides with that in A. And since $a = \neg\neg a$ for all $a \in A_{\neg\neg}$, each element in $A_{\neg\neg}$ is regular, so $A_{\neg\neg}$ is a Boolean algebra. ∎

In light of Proposition 3.8.13, we refer to $A_{\neg\neg}$ as the *Booleanisation* of A (see [32]).

3.9 Density

We now return to the main development of this chapter. Let S be a sublocale of a locale A. $\bigwedge S$ is the least element of S, and $\uparrow\bigwedge S$ is a closed sublocale which clearly contains S. Let $a \in A$, and let $\uparrow a$ contain S. Then

$\bigwedge S \in \uparrow a$, i.e. $a \leqslant \bigwedge S$, and hence $\uparrow\bigwedge S \subseteq \uparrow a$. We can conclude that $\uparrow\bigwedge S$ is the smallest closed sublocale which contains S. We call $\uparrow\bigwedge S$ the *closure* of S, and write $\overline{S} := \uparrow\bigwedge S$. Notice that $\uparrow\bigwedge S = A$ iff $\bigwedge S = 0_A$ iff $0_A \in S$. Therefore,

Definition 3.9.1. A sublocale S of a locale A is said to be *dense* iff $0_A \in S$.

Let X be a topological space, and let Y be a subspace of X. If Y is dense in X, what can we conclude about the induced sublocale $S(Y)$ in $\mathcal{O}X$? Assume Y is dense. We have

$$\begin{aligned}
\mathrm{int}_X(X \setminus Y) &= X \setminus \overline{(X \setminus (X \setminus Y))} \\
&= X \setminus \overline{Y} \\
&= \varnothing.
\end{aligned}$$

Thus, since $S(Y) = \{\mathrm{int}_X((X \setminus Y) \cup V) \mid V \in \mathcal{O}X\}$, it follows that $S(Y)$ is dense in $\mathcal{O}X$ (because $\mathrm{int}_X((X \setminus Y) \cup \varnothing) = \mathrm{int}_X(X \setminus Y) = \varnothing$, so $\varnothing \in S(Y)$). Conversely, if $S(Y)$ is dense in $\mathcal{O}X$ (so that $\varnothing \in S(Y)$), then $\varnothing = \mathrm{int}_X(X \setminus Y)$ and hence Y is dense in X. We summarise this as

Fact 3.9.2. Let Y be a subspace of a topological space X. Then Y is dense in X iff $S(Y)$ is dense in $\mathcal{O}X$.

A topological space may not have a least dense subspace, take for example the real line $\mathbb{R}$ with its usual topology. If $\mathbb{R}$ had a least dense subspace, then any intersection of dense subspaces of $\mathbb{R}$ would also be dense. But both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, while $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \varnothing$. This is where locale theory diverges from point-set topology:

Theorem 3.9.3. *Every locale has a least dense sublocale.*

Proof. Let A be a locale. A glance at Propositions 3.8.5, 3.8.6, and 3.8.8, reveals that $\neg\neg : A \longrightarrow A$ is a nucleus. Thus $A_{\neg\neg}$ is a sublocale, and $0_A \in A_{\neg\neg}$ (so $A_{\neg\neg}$ is dense). Let S be any dense sublocale of A, and let $a \in A_{\neg\neg}$. Then $a = \neg\neg a = (\neg a \Rightarrow 0_A) \in S$. Thus $A_{\neg\neg}$ is the least dense sublocale of A. ∎

Theorem 3.9.3 is referred to as *Isbell's Density Theorem*, it was discovered by Isbell in [11]. We will soon see that any intersection of sublocales of a locale A is also a sublocale of A, thus

Corollary 3.9.4. *Any intersection of dense sublocales of a locale A is again a dense sublocale of A.*

Lemma 3.9.5. *Let X be a topological space, and let $U \in \mathcal{O}X$. Then $\neg\neg U = \mathrm{int}(\overline{U})$.*

Proof. Recall Lemma 3.7.1. We have

$$\begin{aligned}
\neg\neg U &= \neg(U \Rightarrow \varnothing) \\
&= \neg\mathrm{int}(X \setminus U) \\
&= \mathrm{int}(X \setminus U) \Rightarrow \varnothing \\
&= \mathrm{int}(X \setminus \mathrm{int}(X \setminus U)) \\
&= \mathrm{int}(\overline{U}).
\end{aligned}$$

∎

Remark 3.9.6. Recall that an open set $U \in \mathcal{O}X$ is said to be *regular* iff $U = \mathrm{int}(\overline{U})$.

Corollary 3.9.7. *For any topological space X, the set of all regular open sets of X forms a complete Boolean algebra under the inclusion ordering.*

Proof. Take a space X. Then observe that $(\mathcal{O}X)_{\neg\neg} = \{U \in \mathcal{O}X \mid U = \operatorname{int}(\overline{U})\}$. By Proposition 3.8.13, $(\mathcal{O}X)_{\neg\neg}$ is a Boolean algebra, and, since $(\mathcal{O}X)_{\neg\neg}$ is a sublocale, it is a *complete* Boolean algebra. ∎

Isbell's Density Theorem is a source for the construction of non-spatial locales, this is observed as

Proposition 3.9.8. *There exist spatial locales $\mathcal{O}X$ (for X a topological space) for which there are sublocales of $\mathcal{O}X$ that are not induced by subspaces of X.*

Proof. Consider the space $\mathbb{R}$ with its usual topology. Then $(\mathcal{O}\mathbb{R})_{\neg\neg}$ is a sublocale of $\mathcal{O}\mathbb{R}$. As a complete Boolean algebra, $(\mathcal{O}\mathbb{R})_{\neg\neg}$ has no atoms. Indeed, suppose U is an atom in $(\mathcal{O}\mathbb{R})_{\neg\neg}$. Then there exists some $x \in U$, hence $\{x\} \subseteq U$. Since U is an atom, $U = \{x\}$, so $\{x\}$ is a regular open set in $\mathbb{R}$ – but this is a contradiction! Therefore $(\mathcal{O}\mathbb{R})_{\neg\neg}$ is a non-spatial locale, and hence (as a sublocale) it is not induced by a subspace of $\mathbb{R}$. ∎

3.10 The coframe of sublocales, and the frame of nuclei

For any locale A, we will write $\mathcal{S}(A)$ for the set of all sublocales of A, ordered by inclusion.

Theorem 3.10.1. *$\mathcal{S}(A)$ is a coframe (i.e. $\mathcal{S}(A)$ satisfies the order-dual of the frame distributivity law).*

Proof. Take $(S_i)_{i\in I} \subseteq \mathcal{S}(A)$, and put $B = \bigcap_{i\in I} S_i$. Let $M \subseteq B$. Then, for each $i \in I$, $M \subseteq S_i$, hence $\bigwedge M \in S_i$, and thus $\bigwedge M \in B$. It follows that B is closed under meets in A. Take $a \in A$ and $b \in B$. For each $i \in I$, $b \in S_i$, so $(a \Rightarrow b) \in S_i$, and hence $(a \Rightarrow b) \in B$. It follows that B is a sublocale, and $\mathcal{S}(A)$ is closed under arbitrary intersections (i.e. $\mathcal{S}(A)$ is a complete lattice with meet given by intersection). Let $S, T \in \mathcal{S}(A)$. We claim that $S \vee T = \{s \wedge t \mid s \in S, t \in T\}$. Let $R = \{s \wedge t \mid s \in S, t \in T\}$. Let $s_i \in S$ and $t_i \in T$ for all $i \in I$. Then $\bigwedge_{i\in I}(s_i \wedge t_i) = (\bigwedge_{i\in I} s_i) \wedge (\bigwedge_{i\in I} t_i) \in R$. So R is closed under meets in A. For $s \in S$, $t \in T$, and $a \in A$, $a \Rightarrow (s \wedge t) = (a \Rightarrow s) \wedge (a \Rightarrow t) \in R$. It follows that R is a sublocale of A. Since $1_A \in S$ and $1_A \in T$, $s = s \wedge 1_A \in R$ and $t = 1_A \wedge t \in R$. That is, $S \subseteq R$ and $T \subseteq R$. Let $S' \in \mathcal{S}(A)$ be such that $S \subseteq S'$ and $T \subseteq S'$. Then, for all $s \in S$ and $t \in T$, $s \wedge t \in S'$ (since S' is closed under meets in A). We can conclude that R is the least upper bound of $\{S, T\}$ in $\mathcal{S}(A)$. Since $T \vee \bigcap_{i\in I} S_i \subseteq \bigcap_{i\in I}(T \vee S_i)$ holds trivially in $\mathcal{S}(A)$, we need only verify that $\bigcap_{i\in I}(T \vee S_i) \subseteq T \vee \bigcap_{i\in I} S_i$. We have

$$\begin{aligned}\bigcap_{i\in I}(T \vee S_i) &= \{x \in A \mid (\forall i \in I)\ x \in T \vee S_i\}\\ &= \{x \in A \mid (\forall i \in I)(\exists t \in T)(\exists s_i \in S_i)\ x = t \wedge s_i\}.\end{aligned}$$

Let $a \in \bigcap_{i\in I}(T \vee S_i)$. For all $i \in I$, there exists $t \in T$ and $s_i \in S_i$ such that $a = t \wedge s_i$. In fact we can set $t = \bigwedge\{t' \in T \mid x \leqslant t'\}$ and $s_i = \bigwedge\{s_i' \in S_i \mid x \leqslant s_i'\}$, in this way t is held fixed for all $i \in I$. We have

$$\begin{aligned}a &= t \wedge a\\ &= t \wedge (t \Rightarrow a) \quad \text{(recall Proposition 1.1.24)}\\ &= t \wedge (t \Rightarrow (t \wedge s_i))\\ &= t \wedge (t \Rightarrow t) \wedge (t \Rightarrow s_i)\\ &= t \wedge (t \Rightarrow s_i) \in T \vee S_i.\end{aligned}$$

Thus, we have shown that $(t \Rightarrow a) \in S_i$ for all $i \in I$ (i.e. $(t \Rightarrow a) \in \bigcap_{i\in I} S_i$), and hence $a \in T \vee \bigcap_{i\in I} S_i$. We can conclude that $\mathcal{S}(A)$ is a coframe. ∎

Recall that, for a locale A, we wrote $N(A)$ for the set of nuclei on A. We now consider $N(A)$ as a poset with the pointwise order; that is, for $n, n' \in N(A)$, $n \leqslant n' \iff (\forall a \in A)\ n(a) \leqslant n(a')$.

Proposition 3.10.2. *For any locale A, $(\mathcal{S}(A))^{\mathrm{op}}$ is order-isomorphic to $N(A)$.*

Proof. Since a sublocale of A is precisely defined as a fixset of a nucleus, and any two distinct nuclei yield distinct fixsets, it immediately follows that the sublocales of A are in bijection with the nuclei on A. Let $n \leqslant n'$ in $N(A)$, and take $a \in A_{n'}$. Since $a = n'(a)$, it follows that $n(a) \leqslant a$. But n is also inflationary, so $a = n(a)$. Thus $A_{n'} \subseteq A_n$. Conversely, suppose that $A_{n'} \subseteq A_n$. For all $a \in A$, $a \leqslant n'(a)$, so $n(a) \leqslant n(n'(a))$. But $n'(a) \in A_n$, so $n(n'(a)) = n'(a)$. Thus $n(a) \leqslant n'(a)$. We have therefore established that, for any two nuclei n, n' on A,

$$n \leqslant n' \iff A_{n'} \subseteq A_n.$$

■

We immediately obtain

Corollary 3.10.3. *For any frame A, $N(A)$ is a frame.*

3.10.1 A representation theorem

We now turn to a discovery, of Funayama, which shows that every frame is representable as a subframe of a complete Boolean algebra (see [16]). This fact was discovered around the year of 1959. We first need two lemmas.

Lemma 3.10.4. *For any frame A and $a \in A$, the nuclei $o(a)$ and $c(a)$ are complementary elements of $N(A)$.*

Proof. Fix a frame A and $a \in A$. Let $b \in A$. Then

$$\begin{aligned} c(a)(b) \wedge o(a)(b) &= (a \vee b) \wedge (a \Rightarrow b) \\ &= (a \wedge (a \Rightarrow b)) \vee (b \wedge (a \Rightarrow b)) \\ &= (a \wedge b) \vee b \quad \text{(recall Proposition 1.1.24)} \\ &= b. \end{aligned}$$

That is, $c(a) \wedge o(a) = \mathrm{id}_A$ – which is clearly the bottom element of $N(A)$. Next, note that if $n, n' \in N(A)$ are such that $x = (n \vee n')(x)$ for some $x \in A$, then $x = n(x)$ and $x = n'(x)$ (because $x = n(x) \vee n'(x)$ implies $n(x) \leqslant x$ and $n'(x) \leqslant x$). Now suppose $b = (c(a) \vee o(a))(b)$. Then $a \vee b = b$ (i.e. $a \leqslant b$) and $a \Rightarrow b = b$. Hence $1 = (a \Rightarrow a) \leqslant (a \Rightarrow b)$, and so $b = 1$. It follows that, for all $b \in A_{c(a) \vee o(a)}$, $b = 1$. In particular, for all $x \in A$, $(c(a) \vee o(a))(x) = 1$. That is, $c(a) \vee o(a)$ is the constant function which sends each element $x \in A$ to the top element $1 \in A$; $c(a) \vee o(a)$ is clearly the top of $N(A)$. ■

For a frame A, consider the map $c : A \longrightarrow N(A)$ which sends each $a \in A$ to its associated closed nucleus $c(a) = a \vee (-)$.

Lemma 3.10.5. *c is an injective frame homomorphism.*

Proof. Take $a, b \in A$. Then, by using the distributivity of A, $c(a \wedge b) = (a \wedge b) \vee (-) = (a \vee (-)) \wedge (b \vee (-)) = c(a) \wedge c(b)$. Also, $c(1) = 1 \vee (-)$ which is the top element of $N(A)$. Thus c preserves all finite meets. Let $M \subseteq A$. Then $c(\bigvee M) = (\bigvee M) \vee (-) = \bigvee\{m \vee (-) \mid m \in M\}$; this is the join of the $c(m)$ $(m \in M)$ in $N(A)$. Next, if $c(a) = c(b)$, then $a \vee x = b \vee x$ for all $x \in A$. In particular, $a = a \vee 0 = b \vee 0 = b$. We can conclude that $c : A \longrightarrow N(A)$ is an injective frame homomorphism. ■

Now we can state the representation theorem:

Theorem 3.10.6. *Every frame is isomorphic to a subframe of a complete Boolean algebra.*

Proof. Let A be a frame. Then $N(A)$ is a frame, and we can consider the complete Boolean algebra $(N(A))_{\neg\neg}$. Now, the injective frame homomorphism $c : A \longrightarrow N(A)$ can have its codomain restricted to $(N(A))_{\neg\neg}$, since (for all $a \in A$) $\neg\neg c(a) = \neg o(a) = c(a)$; i.e. $c(a)$ is a regular element of $N(A)$ (recall Lemma 3.10.4). Thus we can consider c as an injective map $c : A \rightarrowtail (N(A))_{\neg\neg}$ – and since it still preserves arbitrary joins and finite meets, it is still a frame homomorphism. Thus A is isomorphic to $c[A] \subseteq (N(A))_{\neg\neg}$. ∎

Chapter 4

Freely generated frames

From the algebraic point of view, an important result about frames is that free frames exist, that is to say the forgetful functor **Frm** $\longrightarrow$ **Set** has a left adjoint (see [17]). Typical algebraic structures may described by 'generators and relations' – we will take a similar approach for frames as was introduced by Johnstone in the early eighties (see [12, 13]). The idea is to start with a meet-semilattice A of 'generators', and then introduce a function C on A which assigns to each $a \in A$ a set $C(a)$ of subsets of $\downarrow a$ satisfying a 'stability condition'. Such a function C is known as a *coverage* and is meant to capture the 'relations' imposed on the generators. Every meet-semilattice equipped with a coverage *freely generates* (as defined later in this chapter) a frame in a canonical way – the free frame over a meet-semilattice is just a special case of this. The core references used in this chapter are [12, 13, 31, 32].

4.1 Free frames

In the previous chapter, we claimed that **Frm** is algebraic. **Frm** is clearly equationally presentable [13, 32] – just as semilattices and lattices are equationally presentable – therefore we need to prove that the forgetful functor **Frm** $\longrightarrow$ **Set** is a right adjoint. We will do this in two steps; first we will build free semilattices (over sets), and then construct free frames over meet-semilattices, so that taking the composite we obtain free frames over sets.

Theorem 4.1.1. *The category* **JSLat** *of join-semilattices (and their homomorphisms) is algebraic. Specifically, the free join-semilattice on a set X is the poset FX of all finite subsets of X with the semilattice operation being union.*

Proof. (This proof is adapted from [13]). **JSLat** is indeed equationally presentable; therefore we need to show that the forgetful functor **JSLat** $\longrightarrow$ **Set** is a right adjoint. Let X be a set. Put $FX := \{S \subseteq X \mid S \text{ is finite}\}$. Order FX by inclusion, so that FX becomes a poset. Since a finite union of finite sets is itself a finite set, and $\varnothing \in FX$, it follows that FX is a sub-join-semilattice of $\mathcal{P}(X)$. Let $\eta_X : X \longrightarrow FX$ be the function $x \longmapsto \{x\}$. Let L be a join-semilattice, and let $f : X \longrightarrow L$ be a function. Then f extends uniquely to the join-semilattice homomorphism $\overline{f} : FX \longrightarrow L$ defined by $S \longmapsto \bigvee_L \{f(x) \mid x \in S\}$ for which the triangle

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & FX \arrow[dl, dashed, "\overline{f}"] \\ L & \end{tikzcd}$$

commutes; indeed $(\overline{f} \cdot \eta_X)(x) = \overline{f}(\eta_X(x)) = \overline{f}(\{x\}) = \bigvee_L \{f(x)\} = f(x)$. It follows that the forgetful functor **JSLat** $\longrightarrow$ **Set** is a right adjoint, and FX is the free join-semilattice on X. ■

Corollary 4.1.2. *The free meet-semilattice on a set X is the set $F'X$ of all finite subsets of X ordered by reverse inclusion (i.e. $S \leqslant T$ iff $T \subseteq S$). The finite meet operation in $F'X$ is given by finite union.*

Next we observe that it is relatively straightforward to build free frames over meet-semilattices:

Theorem 4.1.3. *The free frame over a meet-semilattice A is the set $\mathfrak{D}(A)$ of down-sets of A ordered by inclusion.*

Proof. We first need to show that the forgetful functor **Frm** $\longrightarrow$ **MSLat** is a right adjoint. Let A be a meet-semilattice, and put $\mathfrak{D}(A) := \{X \subseteq A \mid X \text{ is a down-set}\}$. Then it is easy to see that an arbitrary union of down-sets is a down-set, and an arbitrary intersection of down-sets is a down-set, making $\mathfrak{D}(A)$ a sub-complete-lattice of $\mathcal{P}(A)$. In particular $\mathfrak{D}(A)$ is a frame (the finite intersection distributes over arbitrary union). Let $\eta_A : A \longrightarrow \mathfrak{D}(A)$ be the function $a \longmapsto {\downarrow} a$. Then $\eta_A(1) = {\downarrow} 1 = A$ (so η_A preserves the top), and $\eta_A(a \wedge a') = {\downarrow}(a \wedge a') = \{x \in A \mid x \leqslant a \wedge a'\} = \{x \in A \mid x \leqslant a \text{ and } x \leqslant a'\} = ({\downarrow} a) \cap ({\downarrow} a') = \eta_A(a) \cap \eta_A(a')$ (so η_A preserves binary meets). It follows that η_A preserves all finite meets and is hence a meet-semilattice homomorphism. Let B be a frame, and let $f : A \longrightarrow B$ be a meet-semilattice homomorphism. We wish to construct a unique frame homomorphism $\overline{f} : \mathfrak{D}(A) \longrightarrow B$ for which $f = \overline{f} \cdot \eta_A$. Define a map $\overline{f} : \mathfrak{D}(A) \longrightarrow B$ by $\overline{f}(X) = \bigvee_{x \in X} f(x)$. We claim that $\overline{f}$ is a frame homomorphism:

- $\overline{f}(\varnothing) = \bigvee f[\varnothing] = \bigvee \varnothing = 0_B$. Thus $\overline{f}$ preserves the bottom.
- $\overline{f}(A) = \bigvee f[A]$. Since f is a meet-semilattice homomorphism, we have $1_B \in f[A]$, and hence $\bigvee f[A] = 1_B$. Thus $\overline{f}$ preserves the top.
- Let $X \subseteq Y$ in $\mathfrak{D}(A)$. Then $f[X] \subseteq f[Y]$ in $\mathcal{P}(B)$, and hence $\bigvee_{x \in X} f(x) \leqslant \bigvee_{y \in Y} f(y)$ in B (i.e. $\overline{f}(X) \leqslant \overline{f}(Y)$). Thus $\overline{f}$ is monotone.
- Take $(X_i)_{i \in I} \subseteq \mathfrak{D}(A)$. For all $j \in I$, we have $X_j \subseteq \bigcup_{i \in I} X_i$ in $\mathfrak{D}(A)$. So, for all $j \in I$, $\overline{f}(X_j) \leqslant \overline{f}(\bigcup_{i \in I} X_i)$ in B, and hence $\bigvee_{i \in I} \overline{f}(X_i) \leqslant \overline{f}(\bigcup_{i \in I} X_i)$. Let $x \in \bigcup_{i \in I} X_i$. Then $x \in X_j$ for some $j \in I$; fix such $j \in I$. We have $f(x) \leqslant \bigvee\{f(z) \mid z \in X_j\} = \overline{f}(X_j) \leqslant \bigvee_{i \in I} \overline{f}(X_i)$. Hence $\bigvee\{f(y) \mid y \in \bigcup_{i \in I} X_i\} \leqslant \bigvee_{i \in I} \overline{f}(X_i)$, i.e. $\overline{f}(\bigcup_{i \in I} X_i) \leqslant \bigvee_{i \in I} \overline{f}(X_i)$. We can conclude that $\overline{f}(\bigcup_{i \in I} X_i) = \bigvee_{i \in I} \overline{f}(X_i)$ and $\overline{f}$ preserves arbitrary joins.
- Take $J, K \in \mathfrak{D}(A)$. We have $\overline{f}(J \cap K) = \bigvee\{f(x) \mid x \in J \cap K\}$. Notice that $\{f(x) \mid x \in J \cap K\} \subseteq \{f(j \wedge k) \mid j \in J, k \in K\}$ (if $x \in J \cap K$, then $x \in J$ and $x \in K$, and $x = x \wedge x$). Let $j \in J$ and $k \in K$. Then $j \wedge k \in J$ and $j \wedge k \in K$ (since J, K are down-sets), so $j \wedge k \in J \cap K$. Thus $f(j \wedge k) \in \{f(x) \mid x \in J \cap K\}$. It follows that $\{f(x) \mid x \in J \cap K\} = \{f(j \wedge k) \mid j \in J, k \in K\}$. Then

$$\begin{aligned}\overline{f}(J \cap K) &= \bigvee\{f(j \wedge k) \mid j \in J, k \in K\} \\ &= \bigvee\{f(j) \wedge f(k) \mid j \in J, k \in K\} \\ &= (\bigvee_{j \in J} f(j)) \wedge (\bigvee_{k \in K} f(k)) \quad \text{(since } B \text{ is a frame)} \\ &= \overline{f}(J) \wedge \overline{f}(K).\end{aligned}$$

  We can conclude that $\overline{f}$ preserves binary meets.

It follows that $\overline{f} : \mathfrak{D}(A) \longrightarrow B$ is a frame homomorphism. Next, observe that there exists a unique frame homomorphism $h : \mathfrak{D}(A) \longrightarrow B$ for which the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \\ \mathfrak{D}(A) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

commutes. Indeed, let $h : \mathfrak{D}(A) \longrightarrow B$ be a frame homomorphism such that $f = h \cdot \eta_A$. Let $X \in \mathfrak{D}(A)$. Then

$$\begin{aligned} h(X) &= h(\bigcup\{\downarrow x \mid x \in X\}) \\ &= \bigvee_{x\in X} h(\downarrow x) \\ &= \bigvee_{x\in X} (h \cdot \eta_A)(x) \\ &= \bigvee_{x\in X} f(x) \\ &=: \overline{f}(X), \end{aligned}$$

and we have already verified that the map $\overline{f} : \mathfrak{D}(A) \longrightarrow B$ is a frame homomorphism. Thus $\overline{f}$ is the unique frame homomorphism $\mathfrak{D}(A) \longrightarrow B$ for which the triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \\ \mathfrak{D}(A) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

commutes. It follows that the forgetful functor **Frm** $\longrightarrow$ **MSLat** is a right adjoint, and the free frame on a meet-semilattice A is the frame $\mathfrak{D}(A)$ of down-sets of A. ∎

We can therefore construct free frames over sets by taking the composite

$$\mathbf{Set} \xrightarrow{F'} \mathbf{MSLat} \xrightarrow{\mathfrak{D}} \mathbf{Frm},$$

where F' is the functor obtained from Corollary 4.1.2. However, we will mostly be interested in the construction of free frames over meet-semilattices – since this is particularly straightforward.

Remark 4.1.4. Bénabou was the first mathematician to show that the free frame on any set is always spatial. One should consult [16] for some historical context of this result, and [17] for further details of the free frame construction over a set.

Corollary 4.1.5. *The category* **Frm** *is algebraic.*

Via Corollary 4.1.5, we may use Theorem 3.1.6 applied to frames.

4.2 Coverages and sites

For a set X we will write $\mathcal{P}^2(X) := \mathcal{P}(\mathcal{P}(X))$. Moreover, for a distributive lattice A,

$$\mathfrak{I}(A) := \text{the set of all ideals of } A\text{, ordered by inclusion.}$$

Following Johnstone in [13], we recall the notion of *coverage*:

Definition 4.2.1. Let A be a meet-semilattice. A *coverage* on A is a function $C : A \longrightarrow \mathcal{P}^2(A)$ which assigns to each $a \in A$ a set $C(a)$ of subsets of $\downarrow a$ (i.e. $C(a) \subseteq \mathcal{P}(\downarrow a)$) such that the following *meet-stability property* is satisfied:

$$(\forall a, b \in A)(\forall S \subseteq \downarrow a) \quad [b \leqslant a \quad \text{and} \quad S \in C(a)] \implies \{b \wedge s \mid s \in S\} \in C(b).$$

A meet-semilattice A equipped with a coverage C is called a *site*, denoted by (A, C). When (A, C) is a site, and $a \in A$, we call each $S \in C(a)$ a *cover* of a. This is indeed strange terminology, since if $S \in C(a)$ then $S \subseteq \downarrow a$ and hence S "sits below" a. However, this terminology will be explained shortly through some examples.

Examples of coverages

Example 4.2.2. Let A be a meet-semilattice, and define a function $C : A \longrightarrow \mathcal{P}^2(A)$ by $C(a) := \{\varnothing\}$. Take $a, b \in A$ with $b \leqslant a$ and $S \in C(a)$ (i.e. $S = \varnothing$). Then $\{b \wedge s \mid s \in S\} = \{b \wedge s \mid s \in \varnothing\} = \{x \mid x \in \varnothing\} = \varnothing \in \{\varnothing\} = C(b)$. So C is a coverage on A.

Example 4.2.3. Let A be a meet-semilattice. Define a function $C : A \longrightarrow \mathcal{P}^2(A)$ by $C(a) := \{\downarrow a\}$. Take $a, b \in A$ with $b \leqslant a$ and $S \in C(a)$ (i.e. $S = \downarrow a$). Put $S_b := \{b \wedge s \mid s \in S\} = \{b \wedge s \mid s \leqslant a\}$. Obviously $S_b \subseteq \downarrow b$. Let $x \in \downarrow b$. Then $x \leqslant b \leqslant a$, so $b \wedge x = x \in S_b$. Hence $S_b = \downarrow b \in C(b)$. Thus C is a coverage on A.

Example 4.2.4. Combining the previous two examples, it is easy to see that, for a meet-semilattice A, the function $C : A \longrightarrow \mathcal{P}^2(A)$ defined by $C(a) := \{\varnothing, \downarrow a\}$ is a coverage on A.

Example 4.2.5. Let A be a meet-semilattice. Define a function $C : A \longrightarrow \mathcal{P}^2(A)$ by $C(a) := \mathcal{P}(\downarrow a)$. Then for any $a, b \in A$ with $b \leqslant a$ and $S \in C(a)$ we have $\{b \wedge s \mid s \in S\} \in \mathcal{P}(\downarrow b)$. So C is a coverage on A.

Example 4.2.6. Let A be a frame. Define $C : A \longrightarrow \mathcal{P}^2(A)$ by $C(a) := \{S \subseteq A \mid \bigvee S = a\}$. Then we see that, for each $a \in A$, $C(a)$ is a collection of subsets of $\downarrow a$. Take $a, b \in A$ with $b \leqslant a$ and $S \in C(a)$. Then $\bigvee\{b \wedge s \mid s \in S\} = b \wedge (\bigvee S) = b \wedge a = b$. So $\{b \wedge s \mid s \in S\} \in C(b)$ and it follows that C is a coverage on the frame A. Notice here that the meet-stability property follows as a consequence of the frame distributivity.

Example 4.2.7. Let A be a distributive lattice. Define $C : A \longrightarrow \mathcal{P}^2(A)$ by $C(a) := \{S \subseteq A \mid S \text{ is finite, and } \bigvee S = a\}$. Again, for each $a \in A$, $C(a)$ is a collection of subsets of $\downarrow a$. Take $a, b \in A$ with $b \leqslant a$ and $S \in C(a)$. Now $\{b \wedge s \mid s \in S\}$ is a finite set, and

$$\underbrace{\bigvee\{b \wedge s \mid s \in S\} = b \wedge \bigvee S}_{\text{binary meet distributes over finite join}} = b \wedge a = b.$$

Hence $\{b \wedge s \mid s \in S\} \in C(b)$ and it follows that C is a coverage on the lattice A. We will point out again that here the meet-stability property follows as a consequence of the distributive property of the lattice A.

A comment on the use of the term *cover*

Examples 4.2.6 and 4.2.7 provide some motivation as to why, when we have a site (A, C), the elements $S \in C(a)$ are called covers of a. The elements $S \in C(a)$ are those subsets of A whose join (if it exists) we would like to be a. That is to say we would like

$$\text{“}S \in C(a) \implies \bigvee S = a\text{”}.$$

In this sense it is natural to say that S covers a. However, this is just an intuitive interpretation of covers. In general we will be working in a meet-semilattice, so we don't necessarily have joins at our disposal.

4.2.1 C-ideals

Let (A, C) be a site. Since A is closed under finite meets, we have the option to work with filters on A. However, from the algebraic viewpoint, ideals are of more importance (see [6]), and we are in this section unpacking some of the algebraic properties of frames; namely the existence of free frames. Now we don't have joins to work with, but we do have the coverage C which provides us with those subsets $S \subseteq A$ that we would like to have joins. Hence we cannot define ideals on A, but we can define the concept of *C-ideal.*

Definition 4.2.8. Let (A, C) be a site and $J \subseteq A$. We say that J is a *C-ideal* iff:

(i) J is a down-set, and

(ii) $(\forall a \in A)(\forall S \subseteq {\downarrow} a) \quad [S \in C(a) \quad \text{and} \quad S \subseteq J] \implies a \in J.$

For a site (A, C) we denote the set of all C-ideals of A by $C\text{-Idl}(A)$, and we order $C\text{-Idl}(A)$ by set-theoretic inclusion. Notice that $C\text{-Idl}(A) \subseteq \mathfrak{D}(A)$.

Examples of C-ideals

Example 4.2.9. Let A be a meet-semilattice and let $C : A \longrightarrow \mathcal{P}^2(A)$ be the coverage defined by $C(a) := \{{\downarrow} a\}$. We claim that $C\text{-Idl}(A) = \mathfrak{D}(A)$. Indeed, let $J \subseteq A$ be a down-set and let $a \in A$. Take $S \in C(a)$, so that $S = {\downarrow} a$ and suppose that $S \subseteq J$. Then clearly we have $a \in J$. So that J is a C-ideal (in this case J is non-empty). Notice that if $J = \varnothing$ then, for each $a \in A$ and $S \in C(a)$, $S \nsubseteq J$ (so we don't need to check the second requirement of Definition 4.2.8) and it follows automatically that $J = \varnothing$ is a C-ideal. Hence $\mathfrak{D}(A) \subseteq C\text{-Idl}(A)$. So the C-ideals of A are precisely the down-sets of A.

Example 4.2.10. Let A be a distributive lattice, and let $C : A \longrightarrow \mathcal{P}^2(A)$ be the coverage defined by $C(a) := \{S \subseteq A \mid S \text{ is finite, and } \bigvee S = a\}$. We will show that $C\text{-Idl}(A) = \mathfrak{J}(A)$. Let $J \in C\text{-Idl}(A)$. Then J is a down-set. Consider the bottom $0 \in A$. Since $0 = \bigvee \varnothing$, it follows that $\varnothing \in C(0)$. Now $\varnothing \subseteq J$, hence $0 \in J$. So J is non-empty. Take $x, y \in J$. Then there exists $a \in A$ such that $a = x \vee y$, in which case $\{x, y\} \in C(a)$. Hence $a \in J$. So J is an ideal. Now let $J' \in \mathfrak{J}(A)$. Then J' is a down-set. Take $a' \in A$ and $S \in C(a')$, such that $S \subseteq J$. Now S is finite and $\bigvee S = a'$, so $a' \in J'$ since J' is closed under finite joins. It follows that J' is a C-ideal. So in this case, the C-ideals of A are precisely the ideals of A.

An important proposition

Lemma 4.2.11. *Let A be a meet-semilattice, and consider the down-set frame $\mathfrak{D}(A)$. Take $X, Y \in \mathfrak{D}(A)$. Then, for all $a \in A$,*

$$a \in (X \Rightarrow Y) \quad \text{iff} \quad X \cap {\downarrow} a \subseteq Y.$$

Proof. Let $a \in A$.

($\Rightarrow$) Suppose that $a \in (X \Rightarrow Y)$. Then $a \in \bigcup\{Z \in \mathfrak{D}(A) \mid X \cap Z \subseteq Y\}$. This means that there exists some $Z \in \mathfrak{D}(A)$ such that $X \cap Z \subseteq Y$ and $a \in Z$. Since Z is a down-set, we have ${\downarrow} a \subseteq Z$. Then $X \cap {\downarrow} a \subseteq X \cap Z \subseteq Y$. Hence $X \cap {\downarrow} a \subseteq Y$.

($\Leftarrow$) Suppose that $X \cap \downarrow a \subseteq Y$. Since $a \in \downarrow a$, it follows that $a \in \bigcup\{Z \in \mathfrak{D}(A) \mid X \cap Z \subseteq Y\}$. That is to say, $a \in (X \Rightarrow Y)$. ∎

For a site (A, C) we have seen that $C\text{-Idl}(A) \subseteq \mathfrak{D}(A)$, but we can in fact say more:

Proposition 4.2.12. *Let (A, C) be a site. Then $C\text{-Idl}(A)$ is a sublocale of the down-set frame $\mathfrak{D}(A)$.*

Proof. (i) Let $(J_i)_{i \in I} \subseteq C\text{-Idl}(A)$. Then $\bigcap_{i \in I} J_i$ is an intersection of down-sets, and is thus itself a down-set. Let $a \in A$ and $S \in C(a)$ such that $S \subseteq \bigcap_{i \in I} J_i$. Then $S \subseteq J_i$ for each $i \in I$, and hence $a \in J_i$ for each $i \in I$, which implies that $a \in \bigcap_{i \in I} J_i$. Thus $\bigcap_{i \in I} J_i \in C\text{-Idl}(A)$. It follows that $C\text{-Idl}(A)$ is closed under meets in $\mathfrak{D}(A)$.

(ii) Take $X \in \mathfrak{D}(A)$ and $J \in C\text{-Idl}(A)$. We need to verify that $(X \Rightarrow J) \in C\text{-Idl}(A)$, where $\Rightarrow$ is the Heyting implication on $\mathfrak{D}(A)$. Note that

$$\underbrace{X \Rightarrow J = \bigcup\{Z \in \mathfrak{D}(A) \mid X \cap Z \subseteq J\}}_{\text{we need to verify that this is a } C\text{-ideal}}$$

Firstly, $X \Rightarrow J$ is a union of down-sets, so $X \Rightarrow J$ is a down-set. Let $a \in A$, $S \in C(a)$ and $S \subseteq (X \Rightarrow J)$. We wish to show that $a \in (X \Rightarrow J)$, i.e. we would like to show that $X \cap \downarrow a \subseteq J$. Take $b \in X \cap \downarrow a$. Then $b \leqslant a$, so $S' := \{b \wedge s \mid s \in S\} \in C(b)$. Let $x \in S'$, then there exists $s \in S$ such that $x \leqslant s$, so $x \in \downarrow S$ and hence $S' \subseteq \downarrow S$. Since $S \subseteq (X \Rightarrow J)$, it follows that $\downarrow S \subseteq (X \Rightarrow J)$ (because $\downarrow(X \Rightarrow J) = X \Rightarrow J$ as $X \Rightarrow J$ is a down-set). Hence $S' \subseteq (X \Rightarrow J)$. Let $s \in S$. Then $b \wedge s \in (X \Rightarrow J)$, and so $X \cap \downarrow(b \wedge s) \subseteq J$. Since $b \in X$, it follows that $b \wedge s \in X$, and in particular $b \wedge s \in X \cap \downarrow(b \wedge s)$ – which gives $b \wedge s \in J$. Thus $S' \in J$, from which we conclude that $b \in J$ since J is a C-ideal. Therefore $X \cap \downarrow a \subseteq J$, and $X \Rightarrow J$ is a C-ideal. ∎

Remark 4.2.13. Since $C\text{-Idl}(A)$ is a sublocale of $\mathfrak{D}(A)$, we know that $C\text{-Idl}(A)$ is a frame in which, for any $(J_i)_{i \in I} \subseteq C\text{-Idl}(A)$,

$$\bigwedge_{i \in I} J_i = \bigcap_{i \in I} J_i \quad \text{and} \quad \bigvee_{i \in I} J_i = \bigcap\{J \in C\text{-Idl}(A) \mid \bigcup_{i \in I} J_i \subseteq J\}.$$

Moreover, the inclusion $j : C\text{-Idl}(A) \hookrightarrow \mathfrak{D}(A)$ is a localic map, i.e. the map $j^* : \mathfrak{D}(A) \twoheadrightarrow C\text{-Idl}(A)$ is a surjective frame homomorphism (where $j^* \dashv j$). Specifically, for $X \in \mathfrak{D}(A)$,

$$j^*(X) = \bigcap\{J \in C\text{-Idl}(A) \mid X \subseteq J\}.$$

4.3 Freely generated frames

We now cover the concept of *freely generated frame*. This was introduced by Johnstone in [12, 13]; his goal was to describe frames by 'generators and relations' (just as any other algebraic structure) in which the generators constitute a meet-semilattice A, and the relations are described by a coverage C on A. Particular applications of this idea include the construction of coproducts in **Frm** (see [13, Ch. 2]) and the description of the locale $\mathfrak{L}(\mathbb{R})$ of the real numbers (see [13, Ch. 4]) – Banaschewski expanded on this description of $\mathfrak{L}(\mathbb{R})$ in [1].

Definition 4.3.1. Let (A, C) be a site, and let B be a frame. A meet-semilattice homomorphism

$f : A \longrightarrow B$ is said to *transform covers to joins* (or is a *C-transformer*) iff

$$(\forall a \in A)(\forall S \in C(a)) \quad f(a) = \bigvee f[S].$$

Remark 4.3.2. Intuitively, for a meet-semilattice homomorphism $f : A \longrightarrow B$ (where (A, C) is a site, and B is a frame) to be a C-transformer, we are requiring that "f preserves joins". Recall that, for $S \in C(a)$, intuitively we would like $\bigvee S = a$; so in some sense the requirement $f(a) = \bigvee f[S]$ means "$f(\bigvee S) = \bigvee f[S]$". Of course (A, C) need not have joins; we are just emphasising the intuition here.

Definition 4.3.3. We say that a frame B is *freely generated* by a site (A, C) iff there exists a C-transformer $f : A \longrightarrow B$ with the property that for any frame B' and any C-transformer $f' : A \longrightarrow B'$ there exists a unique frame homomorphism $h : B \longrightarrow B'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"'] & B' \\ B \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

commutes.

Theorem 4.3.4. *For any site (A, C), the frame C-$\mathrm{Idl}(A)$ is freely generated by (A, C).*

Proof. Let (A, C) be a site, and consider the frame C-$\mathrm{Idl}(A)$. Recall that we have a surjective frame homomorphism $j^* : \mathfrak{D}(A) \twoheadrightarrow C\text{-}\mathrm{Idl}(A)$ defined by $X \longmapsto \bigcap\{J \in C\text{-}\mathrm{Idl}(A) \mid X \subseteq J\}$. Consider the meet-semilattice homomorphism $\eta_A : A \longrightarrow \mathfrak{D}(A)$ defined by $\eta_A(a) := {\downarrow} a$. The composite $j^* \cdot \eta_A$ is thus a meet-semilattice homomorphism $A \longrightarrow C\text{-}\mathrm{Idl}(A)$. For $a \in A$ we have

$$\begin{aligned} j^*(\eta_A(a)) &= j^*({\downarrow} a) \\ &= \bigcap\{J \in C\text{-}\mathrm{Idl}(A) \mid {\downarrow} a \subseteq J\} \\ &= \bigcap\{J \in C\text{-}\mathrm{Idl}(A) \mid a \in J\} \\ &=: J_a. \end{aligned}$$

$(j^* \cdot \eta_A)(a) =: J_a$ is the smallest C-ideal J for which $a \in J$. We claim that $j^* \cdot \eta_A : A \longrightarrow C\text{-}\mathrm{Idl}(A)$ is a C-transformer. Let $a \in A$ and $S \in C(a)$. We must show that $J_a = \bigvee_{s \in S} J_s$.

- Fix some $s \in S$, and consider J_s. We have $S \subseteq {\downarrow} a$, so $s \leqslant a$, which yields $s \in J_a$ (since $a \in J_a$ and J_a is a down-set). But J_s is the smallest C-ideal that contains s, hence $J_s \subseteq J_a$. Thus we have shown that, for all $s \in S$, $J_s \subseteq J_a$.

- Let $J' \in C\text{-}\mathrm{Idl}(A)$ be such that, for all $s \in S$, $J_s \subseteq J'$. Then, for all $s \in S$, $s \in J'$, so $S \subseteq J'$. Since J' is a C-ideal, we obtain $a \in J'$. But J_a is the smallest C-ideal containing a, hence $J_a \subseteq J'$.

From what we have shown above, it follows that J_a is the least upper bound of $\{J_s \mid s \in S\}$, i.e. $J_a = \bigvee_{s \in S} J_s$. We can conclude that the map $j^* \cdot \eta_A : A \longrightarrow C\text{-}\mathrm{Idl}(A)$ is a C-transformer. Let B be a frame, and let $f : A \longrightarrow B$ be a C-transformer. We wish to construct a unique frame homomorphism $h : C\text{-}\mathrm{Idl}(A) \longrightarrow B$ for which $f = h \cdot j^* \cdot \eta_A$. Consider the map $\overline{f} : \mathfrak{D}(A) \longrightarrow B$ defined by $\overline{f}(X) = \bigvee_{x \in X} f(x)$ (as in the proof of

Theorem 4.1.3). Then $\overline{f}$ is the unique frame homomorphism $\mathfrak{D}(A) \longrightarrow B$ for which $f = \overline{f} \cdot \eta_A$. We therefore obtain the following diagram in **MSLat**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \\ \mathfrak{D}(A) \arrow[ur, dashed, "\overline{f}"'] \arrow[d, two heads, "j^*"'] & \\ C\text{-Idl}(A). & \end{tikzcd}$$

Let $g : B \longrightarrow \mathfrak{D}(A)$ be the right adjoint of $\overline{f}$ (g is of course a localic map). Then, for each $b \in B$,

$$\begin{aligned} g(b) &= \bigcup\{X \in \mathfrak{D}(A) \mid \bigvee_{x \in X} f(x) \leqslant b\} \\ &= \bigcup\{X \in \mathfrak{D}(A) \mid (\forall x \in X)\ f(x) \leqslant b\}. \end{aligned}$$

Notice that $g(b)$ is a union of down-sets and is thus itself a down-set. For each $b \in B$, put $A_b := \{x \in A \mid f(x) \leqslant b\}$. Fix some $b \in B$. We claim that $g(b) = A_b$:

- Let $a \in A_b$. Then $f(a) \leqslant b$. Moreover, for all $y \in {\downarrow} a$, we have $f(y) \leqslant f(a)$, and hence $f(y) \leqslant b$. Therefore ${\downarrow} a \in \{X \in \mathfrak{D}(A) \mid (\forall x \in X)\ f(x) \leqslant b\}$. Hence $a \in g(b)$, i.e. $A_b \subseteq g(b)$.

- Let $a \in g(b)$. Then there exists $X \in \mathfrak{D}(A)$ with $a \in X$ and $(\forall x \in X)\ f(x) \leqslant b$. In particular, $f(a) \leqslant b$. Hence $a \in A_b$, i.e. $g(b) \subseteq A_b$.

We can conclude that, for each $b \in B$,

$$g(b) = \{x \in A \mid f(x) \leqslant b\}.$$

We claim that, for all $b \in B$, $g(b)$ is a C-ideal of A. Fix some $b \in B$.

- We have already seen that $g(b)$ is a down-set.

- Let $a \in A$, $S \in C(a)$ and $S \subseteq g(b)$. For all $s \in S$, $f(s) \leqslant b$, and hence $\bigvee f[S] \leqslant b$. Now f is a C-transformer, so we have $f(a) \leqslant b$, and therefore $a \in g(b)$.

We can conclude that $g(b) \in C\text{-Idl}(A)$. Denote by $\widetilde{g}$ the map $B \longrightarrow C\text{-Idl}(A)$ where $(\forall b \in B)\ b \longmapsto g(b)$. Denote by $\overline{f}_{|C\text{-Idl}(A)}$ the restriction of $\overline{f}$ to $C\text{-Idl}(A)$. Both $\widetilde{g}$ and $\overline{f}_{|C\text{-Idl}(A)}$ are monotone, and $\overline{f}_{|C\text{-Idl}(A)} \dashv \widetilde{g}$. Since the sublocale $g[B]$ is contained in the sublocale $C\text{-Idl}(A)$, it follows that $g[B]$ is a sublocale of $C\text{-Idl}(A)$. Therefore $\widetilde{g}$ is a localic map $B \longrightarrow C\text{-Idl}(A)$ (i.e. $\overline{f}_{|C\text{-Idl}(A)}$ is a frame homomorphism). Then we obtain the following commutative diagram in **Loc**$*$:

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "\widetilde{g}"'] & & \mathfrak{D}(A) \\ & C\text{-Idl}(A). \arrow[ur, hook, "j"'] & \end{tikzcd} \tag{4.1}$$

By dualising the diagram (4.1), we obtain the following commutative diagram in **Frm**:

$$\begin{tikzcd} \mathfrak{D}(A) \arrow[rr, "\overline{f}"] \arrow[dr, two heads, "j^*"'] & & B \\ & C\text{-Idl}(A). \arrow[ur, "\overline{f}_{|C\text{-Idl}(A)}"'] & \end{tikzcd} \tag{4.2}$$

Let $t : C\text{-Idl}(A) \longrightarrow B$ be a frame homomorphism for which $\overline{f} = t \cdot j^*$. Then

$$\begin{aligned} \overline{f}_{|C\text{-Idl}(A)} \cdot j^* &= t \cdot j^*, \\ \text{and hence} \quad \overline{f}_{|C\text{-Idl}(A)} &= t \quad (\text{since } j^* \text{ is epic in } \mathbf{Frm}). \end{aligned}$$

Therefore $\overline{f}_{|C\text{-Idl}(A)}$ is the unique frame homomorphism $C\text{-Idl}(A) \longrightarrow B$ for which the diagram (4.2) commutes. In summary, the diagram (viewed as in **MSLat**)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \\ \mathfrak{D}(A) \arrow[ur, dashed, "\overline{f}"'] \arrow[d, two heads, "j^*"'] & \\ C\text{-Idl}(A) \arrow[uur, dashed, bend right, "\overline{f}_{|C\text{-Idl}(A)}"'] & \end{tikzcd}$$

commutes. Let $q : C\text{-Idl}(A) \longrightarrow B$ be a frame homomorphism for which $f = q \cdot j^* \cdot \eta_A$. Then $(q \cdot j^*) \cdot \eta_A = \overline{f} \cdot \eta_A$. Recall that $\overline{f}$ is the unique frame homomorphism $h : \mathfrak{D}(A) \longrightarrow B$ for which $f = h \cdot \eta_A$. Therefore, $q \cdot j^* = \overline{f} = \overline{f}_{|C\text{-Idl}(A)} \cdot j^*$. But j^* is epic in **Frm**, hence $q = \overline{f}_{|C\text{-Idl}(A)}$. Therefore $\overline{f}_{|C\text{-Idl}(A)}$ is the unique frame homomorphism for which the triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j^* \cdot \eta_A"'] & B \\ C\text{-Idl}(A) \arrow[ur, dashed, "\overline{f}_{|C\text{-Idl}(A)}"'] & \end{tikzcd}$$

commutes. We can conclude that the frame $C\text{-Idl}(A)$ is freely generated by the site (A, C). ■

Remark 4.3.5. We have seen that when A is a distributive lattice, we obtain, by setting $(\forall a \in A)\ C(a) := \{S \subseteq A \mid S \text{ is finite}, \bigvee S = a\}$, a site (A, C) for which $C\text{-Idl}(A) = \mathfrak{J}(A) := \{J \subseteq A \mid J \text{ is an ideal of } A\}$. We may therefore conclude that the frame of ideals $\mathfrak{J}(A)$ is freely generated by (A, C). Moreover, as a consequence of Theorem 4.3.4, Johnstone (in [13]) demonstrates that the construction $A \longmapsto \mathfrak{J}(A)$ extends to a functor $\mathfrak{J} : \mathbf{DLat} \longrightarrow \mathbf{Frm}$ (where **DLat** is the category of distributive lattices and their homomorphisms) which is left adjoint to the forgetful functor $\mathbf{Frm} \longrightarrow \mathbf{DLat}$ (the functor $\mathfrak{J}$ is said to be the *ideal functor* and has been studied thoroughly, for example see [36]). Following Banaschewski (in [2]) those frames B which are isomorphic to a $\mathfrak{J}(A)$ (for A a distributive lattice) are said to be *coherent*. Banaschewski showed that all coherent frames are spatial – in fact the spatialness of coherent frames is equivalent to the Boolean Ultrafilter Theorem.

Chapter 5

Compactness

In point-set topology, compactness[1] is a fundamental invariant. It is a property that is often viewed as being internal to a space (when one uses the open-cover definition), but it also has a categorical (or extrinsic) characterisation; the so-called Kuratowski-Mrówka Theorem for compactness:

Theorem 5.0.1. *A space X is compact iff, for all spaces Y, the projection $\pi_Y : X \times Y \longrightarrow Y$ is closed.*

Theorem 5.0.1 was proved in full generality for topological spaces by Mrówka in the year 1959 (see [24]). We note that before this result of Mrówka, there was Kuratowski (in 1931) who proved that for a compact metric space X the projection $\pi_Y : X \times Y \longrightarrow Y$ is closed for any metric space Y [18]. Mrówka's insight was purely topological, however the same result can be proved using the technique of *proper maps*, as we will spell out below.

For the following argumentation, we recall [5]. We call a continuous map $f : X \longrightarrow Y$ of topological spaces *proper* iff f is closed and in every pullback diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

the map f' is closed. We speak of the proper map f being *stably* closed. Not every closed map of spaces is stably closed, and thus 'stably closed' is stronger than 'closed'. Opting for an alternative (yet what will turn out to be equivalent) definition of compactness, we have

Definition 5.0.2. A space X is said to be *compact* iff the unique map $!_X : X \longrightarrow 1$ (from X to the terminal space 1) is proper.

Note that pullbacks of $!_X : X \longrightarrow 1$ are precisely the projections of the form $\pi_Y : X \times Y \longrightarrow Y$ (for $Y \in \mathbf{Top}$); it follows that if $!_X$ is proper then π_Y is closed for all Y, conversely if π_Y is closed for all Y then (since $!_X$ is closed) $!_X$ must be *stably* closed. We therefore obtain Theorem 5.0.1 immediately from Definition 5.0.2, and thus the open-cover definition of compactness and Definition 5.0.2 are equivalent.

The open-cover definition of compactness carries over seamlessly from spaces to frames and locales, and the full Kuratowski-Mrówka Theorem characterising compact locales was proved by Pultr and Tozzi in [35]. Vermeulen

[1] A topological space X is *compact* iff every open cover of X contains a finite subcover.

(in [41]) also provided a proof of the point-free Kuratowski-Mrówka Theorem in which he used (and pioneered) the techniques of proper maps of locales.

The goal of this final chapter of the thesis is to prove the Kuratowski-Mrówka Theorem for frames and locales, while unpacking detailed arguments from [31, 35]; however, we won't use the technique of proper maps of locales.

5.1 Introducing localic compactness

Definition 5.1.1. Let A be a frame. A *cover* of A is a subset $U \subseteq A$ such that $\bigvee U = 1$. A *subcover* U' of a cover U is a subset $U' \subseteq U \subseteq A$ such that $\bigvee U' = 1$.

Definition 5.1.2. A frame A is said to be *compact* iff every cover of A contains a finite subcover.

In the terminology of [33], compactness is a *conservative* property:

Observation 5.1.3. A space X is compact iff its frame of opens $\mathcal{O}X$ is compact.

Lemma 5.1.4. *Let A be a frame, and let $S \subseteq A$ be a sublocale. If S is closed, then the non-empty joins in S coincide with the non-empty joins in A.*

Proof. Let S be closed. Take a non-empty subset $(a_i)_{i\in I} \subseteq S$. Recall that the join $\bigsqcup_{i\in I} a_i$ of the a_i in the sublocale S is given by the formula

$$\bigsqcup_{i\in I} a_i = \bigwedge\{s \in S \mid \bigvee_{i\in I} a_i \leqslant s\}.$$

Since S is closed, $S = \uparrow a$ for some $a \in A$. So,

$$\bigsqcup_{i\in I} a_i = \bigwedge\{s \in A \mid a \leqslant s \text{ and } \bigvee_{i\in I} a_i \leqslant s\}.$$

Now, $a \leqslant a_i$ for all $i \in I$, so we can write

$$\bigsqcup_{i\in I} a_i = \bigwedge\{s \in A \mid \bigvee_{i\in I} a_i \leqslant s\} = \bigvee_{i\in I} a_i.$$

∎

Remark 5.1.5. Even if S is closed, we cannot expect the empty join in S to coincide with the empty join in A. When S is closed we have $S = \uparrow a$ for some $a \in A$, hence $\bigsqcup \varnothing = a$, while $\bigvee \varnothing = 0_A$.

The following two propositions are analogous to results that appear in point-set topology.

Proposition 5.1.6. *Let A be a compact frame, and let $S \subseteq A$ be a closed sublocale. Then S is compact.*

Proof. Firstly, note that if $S = \{1_A\}$ then S is trivially compact. So let S be a non-trivial closed sublocale (i.e. $S = \uparrow a$ for some $a \in A$ with $a \neq 1_A$), and suppose $U \subseteq S$ is a cover of S. Then clearly U is non-empty, hence (by Lemma 5.1.4) U is also a cover of A. Since A is compact, there exists a non-empty finite $U' \subseteq U$ such that $\bigsqcup U' = \bigvee U' = 1_A$. Therefore S is compact. ∎

Proposition 5.1.7. *Let $f : A \longrightarrow B$ be a localic map, and let $S \subseteq A$ be a sublocale. If S is compact, then $f[S]$ is compact.*

Proof. Let S be compact. The localic map $f : A \longrightarrow B$ restricts to a surjective localic map $\overline{f} : S \twoheadrightarrow f[S]$ where $s \longmapsto f(s)$. Let $\overline{f}^* : f[S] \rightarrowtail S$ be the left adjoint of $\overline{f}$. $\overline{f}^*$ is an injective frame homomorphism. Since $\overline{f}$ and $\overline{f}^*$ are adjoint we have $\overline{f} \cdot \overline{f}^* \cdot \overline{f} = \mathrm{id}_{f[S]} \cdot \overline{f}$, and $\overline{f}$ is epic in the category of posets, so $\overline{f} \cdot \overline{f}^* = \mathrm{id}_{f[S]}$. Let $U \subseteq f[S]$ be a cover of $f[S]$. We have $\bigvee \overline{f}^*[U] = \overline{f}^*(\bigvee U) = \overline{f}^*(1) = 1$, i.e. $\overline{f}^*[U]$ is a cover of S. Since S is compact, there exists a finite $U' \subseteq U$ for which $\bigvee \overline{f}^*[U'] = 1$. We have $1 = \overline{f}^*(\bigvee U')$, so $\overline{f}(1) = 1 = \overline{f} \cdot \overline{f}^*(\bigvee U') = \bigvee U'$. We can conclude that $f[S]$ is compact. ■

Proposition 5.1.7 allows us to conclude that compactness is a 'localic property (invariant)' (just as compactness is a topological property): if A is a compact frame (locale), and B is a frame (locale) which is isomorphic to A, then B is compact.

Other than compactness being a localic invariant, compactness of frames is very well-behaved under coproducts of frames (i.e. products of locales): if $(A_i)_{i \in I}$ is a family of compact frames, then their coproduct $\bigoplus_{i \in I} A_i$ is compact – this is the localic Tychonoff Theorem, its proof was first published by P.T. Johnstone in [12]. What is remarkable about this result is that its proof does not require the Axiom of Choice, and is in fact constructive [19].

5.2 Briefly on binary coproducts of frames

Since the category **Frm** is algebraic, **Frm** has coproducts (dually, **Loc** has products) – recall Theorem 3.1.6. We will only be interested in *binary* coproducts of frames in this chapter. Let A and B be frames, we will denote their coproduct in **Frm** by $A \oplus B$. Of course, $A \oplus B$ can either be viewed as a coproduct of frames *or* a product of locales, we have (with morphisms as viewed in the category of posets)

$$\begin{tikzcd} A \arrow[r, shift right, "\iota_A"'] \arrow[r, phantom, "\top"] & A \oplus B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] \arrow[r, phantom, "\top"] & B, \arrow[l, shift left, "\iota_B"] \end{tikzcd}$$

where ι_A, ι_B are the frame coproduct inclusions, and π_A, π_B are the localic product projections.

Definition 5.2.1. A *base* for a frame A is a subset $U \subseteq A$ such that, for each $a \in A$,

$$a = \bigvee \{u \in U \mid u \leqslant a\}.$$

For topological spaces X and Y, a base for the product space $X \times Y$ is provided by 'open rectangles' of the form $U \times V = \pi_X^{-1}[U] \cap \pi_Y^{-1}[V]$ where $U \in \mathcal{O}X, V \in \mathcal{O}Y$ and π_X, π_Y are the product projections. Accordingly, in the point-free case, a base for the coproduct frame $A \oplus B$ is provided by elements of the form $\pi_A^*(a) \wedge \pi_B^*(b)$ $(= \iota_A(a) \wedge \iota_B(b))$ [42] – this agrees with the intuition that π_i^* is formal preimage along π_i. We write, for each $a \in A$ and $b \in B$, $a \oplus b \coloneqq \iota_A(a) \wedge \iota_B(b)$. Hence, the system $\{a \oplus b \mid a \in A, b \in B\}$ is a base for $A \oplus B$. Notice that $\iota_A(a) = \iota_A(a) \wedge 1 = \iota_A(a) \wedge \iota_B(1) = a \oplus 1$, similarly $\iota_B(b) = 1 \oplus b$. We therefore use the convention of writing $\iota_A : a \longmapsto a \oplus 1$ and $\iota_B : b \longmapsto 1 \oplus b$. Moreover, if $0_{A \oplus B} \neq a \oplus b$ and $a \oplus b \leqslant a' \oplus b'$, then $a \leqslant a'$ and $b \leqslant b'$ (see [31]).

Lemma 5.2.2. *Let A and B be frames such that $A \oplus B$ is non-trivial (i.e. $0_{A \oplus B} \neq 1_{A \oplus B}$). Then the coproduct inclusions ι_A and ι_B are monomorphisms in* **Frm**.

Proof. Obviously, if $a = 0_A$, then $\iota_A(a) = 0_{A\oplus B}$ (because frame homomorphisms preserve 0). So suppose $a, a' \in A$ such that $a, a' \neq 0$ (i.e. $0 < a$ and $0 < a'$). Since $A \oplus B$ is non-trivial, and ι_A is order-preserving (monotone), we have $0_{A\oplus B} < \iota_A(a)$ and $0_{A\oplus B} < \iota_A(a')$. Assume $\iota_A(a) = \iota_A(a')$. Then $a \oplus 1 = a' \oplus 1$. So, $a \oplus 1 \leqslant a' \oplus 1$ and $a' \oplus 1 \leqslant a \oplus 1$. Hence, $a \leqslant a'$ and $a' \leqslant a$, i.e. $a = a'$. Therefore ι_A is injective (i.e. a monomorphism in **Frm**). The proof for showing that ι_B is injective is similar. ∎

In what follows, we will often express an arbitrary element c of the coproduct frame $A \oplus B$ as a join of the form $c = \bigvee_{i\in I}(a_i \oplus b_i)$, where, for each $i \in I$, $a_i \in A$ and $b_i \in B$. This is always possible since $\{a \oplus b \mid a \in A, b \in B\}$ is a base for $A \oplus B$.

Consider frame homomorphisms $h_1 : A_1 \longrightarrow B_1$ and $h_2 : A_2 \longrightarrow B_2$. We then have a frame homomorphism $h_1 \oplus h_2 : A_1 \oplus A_2 \longrightarrow B_1 \oplus B_2$, called the *coproduct map* of h_1 and h_2, which is obtained via the universal property of coproducts:

$$\begin{tikzcd}
A_1 \arrow[r, "h_1"] \arrow[d, "\iota_{A_1}"'] & B_1 \arrow[d, "\iota_{B_1}"] \\
A_1 \oplus A_2 \arrow[r, dashed, "h_1\oplus h_2"] & B_1 \oplus B_2 \\
A_2 \arrow[r, "h_2"'] \arrow[u, "\iota_{A_2}"] & B_2. \arrow[u, "\iota_{B_2}"']
\end{tikzcd}$$

Moreover, for $a_1 \in A_1$ and $a_2 \in A_2$, we have

$$\begin{aligned}
h_1 \oplus h_2(a_1 \oplus a_2) &= h_1 \oplus h_2(\iota_{A_1}(a_1) \wedge \iota_{A_2}(a_2)) \\
&= h_1 \oplus h_2(\iota_{A_1}(a_1)) \wedge h_1 \oplus h_2(\iota_{A_2}(a_2)) \\
&= \iota_{B_1}(h_1(a_1)) \wedge \iota_{B_2}(h_2(a_2)) \\
&= h_1(a_1) \oplus h_2(a_2).
\end{aligned}$$

5.2.1 The unit decomposition property

Lemma 5.2.3. *Let A and B be frames, such that $A \oplus B$ is non-trivial, and set $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$. Then $\bigvee_{i\in I} a_i = 1_A$ and $\bigvee_{i\in I} b_i = 1_B$.*

Proof. We have $\bigvee_{i\in I}(\iota_A(a_i) \wedge \iota_B(b_i)) = 1_{A\oplus B}$. For each $i \in I$, $\iota_A(a_i) \wedge \iota_B(b_i) \leqslant \iota_A(a_i)$, hence $\bigvee_{i\in I}(\iota_A(a_i) \wedge \iota_B(b_i)) \leqslant \bigvee_{i\in I} \iota_A(a_i)$, and it follows that $\bigvee_{i\in I} \iota_A(a_i) = \iota_A(\bigvee_{i\in I} a_i) = 1_{A\oplus B}$. Since $\iota_A \dashv \pi_A$ (where ι_A is injective, and π_A is surjective) we have $\pi_A \cdot \iota_A = \mathrm{id}_A$. Now, $1_A = \pi_A(1_{A\oplus B}) = \pi_A(\iota_A(\bigvee_{i\in I} a_i)) = \bigvee_{i\in I} a_i$. An identical argument to the above yields $\bigvee_{i\in I} b_i = 1_B$. ∎

The following theorem is a result of Kříž and Pultr in their paper [20, Theorem 3.9], to prove it, they define objects called semitrees and invoke the Axiom of Choice. Below, our proof avoids the use of both semitrees and the Axiom of Choice.

Theorem 5.2.4. *Let A be a compact frame, and let B be a non-trivial frame (i.e. $0_B \neq 1_B$). Set $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$. Then there exists a finite $K \subseteq I$ for which $\bigvee_{i\in K} a_i = 1$ and $\bigwedge_{i\in K} b_i \neq 0$.*

Proof. By Lemma 5.2.3 we have $\bigvee_{i\in I} a_i = 1$ and $\bigvee_{i\in I} b_i = 1$. Since A is compact, there exists a finite $K' \subseteq I$ for which $\bigvee_{i\in K'} a_i = 1$. Since $\bigvee_{i\in I} b_i = 1$, there exists $\alpha \in I$ for which $b_\alpha \neq 0$; fix such $\alpha \in I$. Put $K = K' \cup \{\alpha\}$. Then K is finite, and $\bigvee_{i\in K} a_i = 1$. Suppose $\bigwedge_{i\in K} b_i = 0$. Then $\iota_B(\bigwedge_{i\in K} b_i) = \bigwedge_{i\in K} \iota_B(b_i) = 0_{A\oplus B}$.

For each $i \in K$, $\iota_A(a_i) \wedge \iota_B(b_i) \leqslant \iota_B(b_i)$, thus, for each $i \in K$, $\bigvee_{i\in K}(\iota_A(a_i) \wedge \iota_B(b_i)) \leqslant \iota_B(b_i)$. This yields $\bigvee_{i\in K}(\iota_A(a_i) \wedge \iota_B(b_i)) \leqslant \bigwedge_{i\in K} \iota_B(b_i)$, and hence $\bigvee_{i\in K}(\iota_A(a_i) \wedge \iota_B(b_i)) = 0_{A\oplus B}$. For each $i \in K$, $\iota_A(a_i) \wedge \iota_B(b_\alpha) \leqslant \bigvee_{i\in K}(\iota_A(a_i) \wedge \iota_B(b_i))$. It follows that, for each $i \in K$, $\iota_A(a_i) \wedge \iota_B(b_\alpha) = 0_{A\oplus B}$, hence $\bigvee_{i\in K}(\iota_A(a_i) \wedge \iota_B(b_\alpha)) = 0_{A\oplus B}$. By the frame distributivity we have $0_{A\oplus B} = \iota_B(b_\alpha) \wedge (\bigvee_{i\in K} \iota_A(a_i)) = \iota_B(b_\alpha) \wedge \iota_A(\bigvee_{i\in K} a_i) = \iota_B(b_\alpha) \wedge 1_{A\oplus B}$. It follows that $\iota_B(b_\alpha) = 0_{A\oplus B}$. But since $\iota_B \dashv \pi_B$ (with ι_B injective and π_B surjective), it follows that $\pi_B(\iota_B(b_\alpha)) = b_\alpha = \pi_B(0_{A\oplus B}) = 0_B$ (note that $\pi_B(0_{A\oplus B}) = 0_B$ because π_B is monotone and surjective). We have thus obtained a contradiction. It must therefore be the case that $\bigwedge_{i\in K} b_i \neq 0$. ■

The proof of Theorem 5.2.4 appears as though it invokes a proof by contradiction, but it is in fact a *proof of negation* in which the principle of excluded middle is avoided (for further details on the technique of proof of negation, see [28]). We have therefore obtained Theorem 5.2.4 constructively.

Definition 5.2.5. Let A be a frame. A is said to satisfy the *unit decomposition property* (*UD*) iff, for any frame B and each decomposition of the unit $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$, the system $\{\bigwedge_{i\in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\}$ is a cover of B.

Proposition 5.2.6. *Every compact frame satisfies UD.*

Proof. Let A be a compact frame, B an arbitrary frame, and set $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$. Put $U = \{\bigwedge_{i\in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\}$. Suppose that U is not a cover of B, and put $b = \bigvee U$. Then $b \neq 1$, hence the closed sublocale $\uparrow b$ is a non-trivial frame. The localic embedding $j : \uparrow b \hookrightarrow B$ is right adjoint to the sublocale homomorphism $j^* : B \twoheadrightarrow \uparrow b$. For each $x \in B$ and $y \in \uparrow b$, $j^*(x) \leqslant y$ iff $x \leqslant y$. Hence we can write $j^*(x) = \bigwedge\{y \in \uparrow b \mid x \leqslant y \text{ in } B\} = \bigwedge\{y \in B \mid b \leqslant y \text{ and } x \leqslant y\} = b \vee x$. Consider the frame homomorphism $\mathrm{id}_A \oplus j^* : A \oplus B \longrightarrow A\oplus \uparrow b$. We have $1_{A\oplus\uparrow b} = \mathrm{id}_A \oplus j^*(\bigvee_{i\in I}(a_i \oplus b_i)) = \bigvee_{i\in I} \mathrm{id}_A \oplus j^*(a_i \oplus b_i) = \bigvee_{i\in I}(a_i \oplus j^*(b_i))$. By Theorem 5.2.4 there is a finite $K \subseteq I$ for which $\bigvee_{i\in K} a_i = 1$ and $\bigwedge_{i\in K} j^*(b_i) \neq b$. It follows that $\bigwedge_{i\in K} b_i \in U$, so $\bigwedge_{i\in K} b_i \leqslant b$, and hence $b \vee (\bigwedge_{i\in K} b_i) = b$. On the other hand, $b \neq \bigwedge_{i\in K} j^*(b_i) = j^*(\bigwedge_{i\in K} b_i) = b \vee (\bigwedge_{i\in K} b_i)$. We have thus obtained a contradiction. It must be the case that U is a cover of B, and hence A satisfies UD. ■

Lemma 5.2.7. *Let A be a frame which satisfies UD, and let B be an arbitrary frame. Take $c \in A\oplus B$, and set $c = \bigvee_{i\in I}(a_i \oplus b_i)$. If there exists $b' \in B$ such that $1\oplus b' \leqslant c$, then $b' \leqslant \bigvee\{\bigwedge_{i\in K} b_i \mid K \subseteq I$ for which $\bigvee_{i\in K} a_i = 1\}$.*

Proof. Let $b' \in B$ be such that $1 \oplus b' \leqslant c$. Consider the sublocale homomorphism $h : B \twoheadrightarrow \downarrow b'$ defined by $x \longmapsto b' \wedge x$. Consider the frame homomorphism $\mathrm{id}_A \oplus h : A \oplus B \longrightarrow A\oplus \downarrow b'$. We have $\mathrm{id}_A \oplus h(1 \oplus b') = 1\oplus h(b') = 1\oplus b' = 1_A\oplus 1_{\downarrow b'} = 1_{A\oplus\downarrow b'}$. Thus $1_{A\oplus\downarrow b'} = \mathrm{id}_A\oplus h(c) = \mathrm{id}_A\oplus h(\bigvee_{i\in I}(a_i\oplus b_i)) = \bigvee_{i\in I}(a_i\oplus h(b_i)) = \bigvee_{i\in I}(a_i \oplus (b' \wedge b_i))$. Since A satisfies UD, we have $b' = \bigvee\{\bigwedge_{i\in K}(b' \wedge b_i) \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\} = b' \wedge \bigvee\{\bigwedge_{i\in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\}$. Hence $b' \leqslant \bigvee\{\bigwedge_{i\in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\}$. ■

5.3 Closed localic maps

We now collect together a few important and useful characterisations of closed localic maps.

Proposition 5.3.1. *Let $f : A \longrightarrow B$ be a monotone map between posets. We have $f[\uparrow a] \subseteq \uparrow f(a)$ for each $a \in A$.*

Proof. Let $a \in A$ and $b \in f[\uparrow a]$. There exists an $x \in A$ with $a \leqslant x$ such that $b = f(x)$. Since f is monotone, we have $f(a) \leqslant f(x) = b$. Hence $b \in \uparrow f(a)$. ■

Corollary 5.3.2. *Let $f : A \longrightarrow B$ be a localic map. For each sublocale $S \subseteq A$ we have $f[\overline{S}] \subseteq \overline{f[S]}$.*

Proof. Let $S \subseteq A$ be a sublocale, and set $a = \bigwedge S$. By Proposition 5.3.1 we have $f[\overline{S}] = f[\uparrow \bigwedge S] = f[\uparrow a] \subseteq \uparrow f(a) = \uparrow f(\bigwedge S) = \uparrow \bigwedge f[S] = \overline{f[S]}$. ■

Lemma 5.3.3. *Let $f : A \longrightarrow B$ be a localic map. For each $a \in A$ and $b \in B$, we have $f(a) \vee b \leqslant f(a \vee f^*(b))$.*

Proof. Let $a \in A$ and $b \in B$. Since $f^* \dashv f$, we have $b \leqslant ff^*(b)$. Hence $f(a) \vee b \leqslant f(a) \vee ff^*(b)$. We have $a \leqslant a \vee f^*(b)$ and $f^*(b) \leqslant a \vee f^*(b)$, so $f(a) \leqslant f(a \vee f^*(b))$ and $ff^*(b) \leqslant f(a \vee f^*(b))$. Hence $f(a) \vee ff^*(b) \leqslant f(a \vee f^*(b))$. ■

Lemma 5.3.4. *Let $f : A \longrightarrow B$ be a localic map. For each $a \in A$ and $b, b' \in B$ we have*

$$b' \leqslant f(a) \vee b \implies f^*(b') \leqslant a \vee f^*(b).$$

Proof. Take $a \in A$ and $b, b' \in B$. Let $b' \leqslant f(a) \vee b$. Then $f^*(b') \leqslant f^*f(a) \vee f^*(b)$. Since $f^* \dashv f$, we have $f^*f(a) \leqslant a$. Thus $f^*f(a) \vee f^*(b) \leqslant a \vee f^*(b)$. ■

Definition 5.3.5. A localic map $f : A \longrightarrow B$ is said to be *closed* iff, whenever $S \subseteq A$ is a closed sublocale, the sublocale $f[S]$ is closed.

Proposition 5.3.6. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$, $f[\uparrow a] = \uparrow f(a)$.*

Proof. ($\Rightarrow$) Let f be closed, and take $a \in A$. Then $f[\uparrow a] = \uparrow b$ for some $b \in B$. Since f is monotone, $f(a)$ is the least element of $f[\uparrow a] = \uparrow b$. That is to say, $b = f(a)$.

($\Leftarrow$) Trivial. ■

Combining the results of Propositions 5.3.1 and 5.3.6, we obtain

Corollary 5.3.7. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$, $\uparrow f(a) \subseteq f[\uparrow a]$.*

As in point-set topology, closedness of localic maps can be reformulated in terms of a closure operator:

Corollary 5.3.8. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for any sublocale $S \subseteq A$, $\overline{f[S]} \subseteq f[\overline{S}]$.*

Proposition 5.3.9. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$ and $b \in B$, $f(a \vee f^*(b)) \leqslant f(a) \vee b$.*

Proof. ($\Rightarrow$) Let f be closed, and take $a \in A$ and $b \in B$. We have $f[\uparrow a] = \uparrow f(a)$. Note that $f(a) \vee b \in \uparrow f(a)$, hence there exists $x \in \uparrow a$ such that $f(a) \vee b = f(x)$. Then $b \leqslant f(x)$, so $f^*(b) \leqslant x$. Since $a \leqslant x$, we have $a \vee f^*(b) \leqslant x$, and hence $f(a \vee f^*(b)) \leqslant f(x)$.

($\Leftarrow$) Suppose that, for each $a \in A$ and $b \in B$, $f(a \vee f^*(b)) \leqslant f(a) \vee b$. Take $a \in A$ and $b \in B$ such that $f(a) \leqslant b$ (i.e. $b \in \uparrow f(a)$). Then $f(a) \vee b = b$, so $f(a \vee f^*(b)) \leqslant b$. By Lemma 5.3.3, $b \leqslant f(a \vee f^*(b))$, so $b = f(a \vee f^*(b))$. Hence $b \in f[\uparrow a]$. ■

Corollary 5.3.10. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$ and $b \in B$, $f(a) \vee b = f(a \vee f^*(b))$.*

Proposition 5.3.11. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$ and $b, b' \in B$, $b' \leqslant f(a) \vee b \iff f^*(b') \leqslant a \vee f^*(b)$.*

Proof. f is closed iff $(\forall a \in A)(\forall b \in B)\, f(a) \vee b = f(a \vee f^*(b))$ iff $(\forall a \in A)(\forall b, b' \in B)\, b' \leqslant f(a) \vee b \iff b' \leqslant f(a \vee f^*(b))$ iff $(\forall a \in A)(\forall b, b' \in B)\, b' \leqslant f(a) \vee b \iff f^*(b') \leqslant a \vee f^*(b)$. ■

Corollary 5.3.12. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$ and $b, b' \in B$, $f^*(b') \leqslant a \vee f^*(b) \implies b' \leqslant f(a) \vee b$.*

Proposition 5.3.13. *Let $f : A \longrightarrow B$ be a localic map. f is closed iff, for each $a \in A$, there exists $b'' \in B$ such that $f^*(b'') \leqslant a$ and, for each $b, b' \in B$, $f^*(b') \leqslant a \vee f^*(b) \implies b' \leqslant b'' \vee b$.*

Proof. ($\Rightarrow$) Let f be closed, and take $a \in A$. Since $f^* \dashv f$, $f^*(f(a)) \leqslant a$. Let $b, b' \in B$ be such that $f^*(b') \leqslant a \vee f^*(b)$. Since f is closed, $b' \leqslant f(a) \vee b$.

($\Leftarrow$) Let the conditions of the hypothesis be satisfied. Let $a \in A$ and $b, b' \in B$ be such that $f^*(b') \leqslant a \vee f^*(b)$. There exists $b'' \in B$ for which $f^*(b'') \leqslant a$, hence $b'' \leqslant f(a)$. Then $b' \leqslant b'' \vee b \leqslant f(a) \vee b$. Therefore f is closed. ■

5.3.1 Closed projections

Of the various characterisations for closed localic maps thus presented, Corollary 5.3.12 and Proposition 5.3.13 will be of particular importance to us. Consider a binary coproduct of frames $A \oplus B$, and the localic projection $\pi_B : A \oplus B \longrightarrow B$. Corollary 5.3.12 applied to π_B yields

Observation 5.3.14. π_B is closed iff, for each $c \in A \oplus B$ and $b, b' \in B$,

$$1 \oplus b' \leqslant c \vee (1 \oplus b) \implies b' \leqslant \pi_B(c) \vee b.$$

Meanwhile, Proposition 5.3.13 yields

Observation 5.3.15. π_B is closed iff, for each $c \in A \oplus B$, there exists $b'' \in B$ such that $1 \oplus b'' \leqslant c$ and, for each $b, b' \in B$, $1 \oplus b' \leqslant c \vee (1 \oplus b) \implies b' \leqslant b'' \vee b$.

5.4 Constructing the localic Kuratowski-Mrówka Theorem

We now have all the machinery available to construct the Kuratowski-Mrówka Theorem.

Theorem 5.4.1. *Let A be a frame satisfying UD. For any frame B, the projection $\pi_B : A \oplus B \longrightarrow B$ is closed.*

Proof. Take a frame B, and consider the coproduct $A \oplus B$. Let $c \in A \oplus B$, and set $c = \bigvee_{i \in I}(a_i \oplus b_i)$. Put $b'' = \bigvee\{\bigwedge_{i \in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i \in K} a_i = 1\}$. For each $K \subseteq I$ satisfying $\bigvee_{i \in K} a_i = 1$, we have $a_j \oplus \bigwedge_{i \in K} b_i \leqslant a_j \oplus b_j$ for all $j \in K$. Hence, $\bigvee_{j \in K}(a_j \oplus \bigwedge_{i \in K} b_i) \leqslant \bigvee_{j \in K}(a_j \oplus b_j) \leqslant c$. By frame distributivity, we have $(\bigvee_{j \in K} a_j) \oplus \bigwedge_{i \in K} b_i \leqslant c$, and hence $1 \oplus \bigwedge_{i \in K} b_i \leqslant c$. Therefore $1 \oplus b'' \leqslant c$. Let $b, b' \in B$ be such that $1 \oplus b' \leqslant c \vee (1 \oplus b)$. Then we may write $1 \oplus b' \leqslant (\bigvee_{i \in I}(a_i \oplus b_i)) \vee (1 \oplus b) = \bigvee_{i \in I \cup \{*\}}(a_i \oplus b_i)$ where $a_* = 1$ and $b_* = b$. By Lemma 5.2.7, $b' \leqslant \bigvee\{\bigwedge_{i \in K} b_i \mid K \subseteq I \cup \{*\} \text{ for which } \bigvee_{i \in K} a_i = 1\} = \bigvee\{\bigwedge_{i \in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i \in K} a_i = 1\} \vee b = b'' \vee b$. We may conclude that π_B is closed. ■

Since every compact frame satisfies UD (Proposition 5.2.6) we obtain

Corollary 5.4.2. *Let A be a compact frame. For any frame B, the projection $\pi_B : A \oplus B \longrightarrow B$ is closed.*

We now wish to obtain the converse of Corollary 5.4.2. That is, we wish to show that if $\pi_B : A \oplus B \longrightarrow B$ is closed for all frames B, then A is compact. Our approach will be through the contrapositive. That is, we will assume A to be a non-compact frame, then construct a frame B for which $\pi_B : A \oplus B \longrightarrow B$ is not closed.

5.4.1 The converse

Let A be a non-compact frame. Fix a cover U of A for which, for all finite $F \subseteq U$, F does not cover A. Define $A_U \coloneqq \{M \subseteq A \mid 1 \in M \implies \exists \text{ finite } F \subseteq U \text{ for which } \uparrow\bigvee F \subseteq M\}$.

Proposition 5.4.3. *A_U is an Alexandroff topology on A.*

Proof. As $1 \notin \varnothing$, $\varnothing \in A_U$. We have $1 \in A$ and, for all finite $F \subseteq U$, $\uparrow\bigvee F \subseteq A$. Hence $A \in A_U$. Let $(M_i)_{i\in I} \subseteq A_U$. Suppose that $1 \in \bigcup_{i\in I} M_i$. Then there exists $j \in I$ for which $1 \in M_j$. As $M_j \in A_U$, there exists a finite $F \subseteq U$ for which $\uparrow\bigvee F \subseteq M_j \subseteq \bigcup_{i\in I} M_i$. Hence $\bigcup_{i\in I} M_i \in A_U$. Now instead suppose that $1 \in \bigcap_{i\in I} M_i$. Then, for each $i \in I$, $1 \in M_i$. For each $i \in I$, there exists a finite $F_i \subseteq U$ for which $\uparrow\bigvee F_i \subseteq M_i$. $\bigcap_{i\in I} F_i$ is finite and, since (for each $j \in I$) $\bigcap_{i\in I} F_i \subseteq F_j$, we have (for each $j \in I$) $\uparrow\bigvee(\bigcap_{i\in I} F_i) \subseteq \uparrow\bigvee F_j \subseteq M_j$. Thus $\uparrow\bigvee(\bigcap_{i\in I} F_i) \subseteq \bigcap_{j\in I}(\uparrow\bigvee F_j) \subseteq \bigcap_{j\in I} M_j$. We conclude that $\bigcap_{i\in I} M_i \in A_U$. ■

Corollary 5.4.4. *A_U is both a frame (spatial, in particular) and a coframe.*

Remark 5.4.5. From this point onwards, we will treat A_U as a frame.

Observation 5.4.6. For each $u \in U$, $\uparrow u \in A_U$.

Proof. Take $u \in U$, and consider $\uparrow u$. We have $1 \in \uparrow u$ and, by setting $F_u = \{u\}$, we obtain $\bigvee F_u = u$. Hence $\uparrow\bigvee F_u \subseteq \uparrow u$. ■

Consider the coproduct $A \oplus A_U$, and set $c \coloneqq \bigvee_{u\in U}(u \oplus \uparrow u)$.

Lemma 5.4.7. $1_{A\oplus A_U} = c \vee (1_A \oplus (A \setminus \{1_A\}))$.

Proof. For each $u \in U$, $u \oplus A = u \oplus (\uparrow u \vee (A \setminus \{1\})) = (u \oplus \uparrow u) \vee (u \oplus (A \setminus \{1\})) \leqslant c \vee (1 \oplus (A \setminus \{1\}))$. Hence, $\bigvee_{u\in U}(u \oplus A) \leqslant c \vee (1 \oplus (A \setminus \{1\}))$. But $\bigvee_{u\in U}(u \oplus A) = (\bigvee U) \oplus A = 1_A \oplus 1_{A_U} = 1_{A\oplus A_U}$. ■

Lemma 5.4.8. *Consider the localic projection $\pi_{A_U} : A \oplus A_U \longrightarrow A_U$. We have $\pi_{A_U}(c) = \varnothing$.*

Proof. Suppose that $\pi_{A_U}(c) \neq \varnothing$. If $1 \in \pi_{A_U}(c)$, then there exists a finite $F \subseteq U$ for which $\uparrow\bigvee F \subseteq \pi_{A_U}(c)$. Since $\bigvee F \neq 1$, there exists $a \in A$ with $a \neq 1$ such that $a \in \pi_{A_U}(c)$. If $1 \notin \pi_{A_U}(c)$, then there exists $a \in A$ with $a \neq 1$ such that $a \in \pi_{A_U}(c)$. Thus, fix an $a \in A$ with $a \neq 1$ such that $a \in \pi_{A_U}(c)$. Consider the frame homomorphism $h_a : A_U \longrightarrow A$, defined by

$$h_a(M) \coloneqq \begin{cases} 1 & \text{for } a \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the coproduct diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "\mathrm{id}_A"'] & A \oplus A_U \arrow[d, dashed, "\varphi"] & A_U \arrow[l, "\iota_{A_U}"'] \arrow[dl, "h_a"] \\ & A. & \end{tikzcd}$$

For each $u \in U$, $\varphi(u \oplus \uparrow u) = \varphi(\iota_A(u) \wedge \iota_{A_U}(\uparrow u)) = \varphi(\iota_A(u)) \wedge \varphi(\iota_{A_U}(\uparrow u)) = u \wedge h_a(\uparrow u)$. Now

$$h_a(\uparrow u) = \begin{cases} 1 & \text{for } u \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\varphi(u \oplus \uparrow u) = \begin{cases} u & \text{for } u \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

This means that a is an upper bound of the set $\{\varphi(u \oplus \uparrow u) \mid u \in U\}$, and hence $\bigvee_{u \in U} \varphi(u \oplus \uparrow u) = \varphi(\bigvee_{u \in U}(u \oplus \uparrow u)) = \varphi(c) \leqslant a$. Also, $\varphi(1 \oplus \pi_{A_U}(c)) = \varphi(\iota_A(1) \wedge \iota_{A_U}(\pi_{A_U}(c))) = \varphi(\iota_A(1)) \wedge \varphi(\iota_{A_U}(\pi_{A_U}(c))) = 1 \wedge h_a(\pi_{A_U}(c)) = 1 \wedge 1 = 1$. Thus $\varphi(c) < \varphi(1 \oplus \pi_{A_U}(c))$. We have $\varphi(1 \oplus \pi_{A_U}(c)) \nleqslant \varphi(c)$, which implies that $1 \oplus \pi_{A_U}(c) \nleqslant c$, and hence $\iota_{A_U}(\pi_{A_U}(c)) \nleqslant c$. This is a contradiction, since $\iota_{A_U} \dashv \pi_{A_U}$. We may conclude that $\pi_{A_U}(c) = \varnothing$. ∎

Proposition 5.4.9. *The projection $\pi_{A_U} : A \oplus A_U \longrightarrow A_U$ is not closed.*

Proof. Recall Observation 5.3.14. By Lemma 5.4.7, we have $1_A \oplus 1_{A_U} \leqslant c \vee (1_A \oplus (A \smallsetminus \{1_A\}))$. While, by Lemma 5.4.8, $1_{A_U} \nleqslant \pi_{A_U}(c) \vee (A \smallsetminus \{1_A\})$. ∎

In all we have shown that if A is non-compact, then there exists a frame B for which the projection $\pi_B : A \oplus B \longrightarrow B$ is not closed (i.e. if the projection $\pi_B : A \oplus B \longrightarrow B$ is closed for all frames B, then A is compact). We have thus obtained the point-free Kuratowski-Mrówka Theorem:

Theorem 5.4.10. *Let A be a frame. The following statements on A are equivalent*:

(*i*) *A is compact.*

(*ii*) *A satisfies UD.*

(*iii*) *For all frames B, the localic projection $\pi_B : A \oplus B \longrightarrow B$ is closed.*

Bibliography

[1] B. Banaschewski, *The real numbers in pointfree topology*, vol. 12, Universidad de Coimbra, 1997.

[2] ——, *Coherent frames*, in Continuous Lattices: Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV) Held at the University of Bremen, Germany, November 9–11, 1979, Springer, 2006, pp. 1–11.

[3] G. Birkhoff, *Lattice Theory*, vol. 25, American Mathematical Soc., 1940.

[4] T.-D. Bradley, T. Bryson, and J. Terilla, *Topology: A Categorical Approach*, The MIT Press, 2020.

[5] M. M. Clementino, E. Giuli, and W. Tholen, *A Functional Approach to General Topology*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2003, p. 102–164.

[6] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 2nd ed., 2002.

[7] C. H. Dowker and D. Papert, *Quotient frames and subspaces*, Proceedings of the London Mathematical Society, 3 (1966), pp. 275–296.

[8] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, Cambridge University Press, 3rd ed., 2018.

[9] F. Hausdorff, *Grundzüge der Mengenlehre*, vol. 7, von Veit, 1914.

[10] ——, *Set Theory*, vol. 119, American Mathematical Soc., 2021.

[11] J. R. Isbell, *Atomless parts of spaces*, Mathematica Scandinavica, 31 (1972), pp. 5–32.

[12] P. T. Johnstone, *Tychonoff's theorem without the axiom of choice*, Fund. Math., 113 (1981), pp. 21–35.

[13] ——, *Stone spaces*, Cambridge University Press, 1982.

[14] ——, *The point of pointless topology*, Bulletin of the American Mathematical Society, 8 (1983), pp. 41–53.

[15] ——, *The Art of Pointless Thinking: A Student's Guide to the Category of Locales*, Category Theory at Work, (1991).

[16] ——, *Elements of the History of Locale Theory*, in Handbook of the history of general topology, Springer, 2001, pp. 835–851.

[17] ——, *Sketches of an Elephant: A Topos Theory Compendium*, vol. 2, Oxford University Press, 2002.

[18] K. Kuratowski, *Evaluation de la classe borélienne ou projective d'un ensemble de points à l'aide des symboles logiques*, Fundamenta Mathematicae, 17 (1931), pp. 249–272.

[19] I. Kříž, *A constructive proof of the Tychonoff's theorem for locales*, Commentationes Mathematicae Universitatis Carolinae, 26 (1985), pp. 619–630.

[20] I. Kříž and A. Pultr, *Peculiar behavior of connected locales*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 30 (1989), pp. 25–43.

[21] T. Leinster, *Basic Category Theory*, Cambridge University Press, 2014.

[22] A. McCluskey and B. McMaster, *Undergraduate Topology: A Working Textbook*, OUP Oxford, 2014.

[23] G. H. Moore, *The emergence of open sets, closed sets, and limit points in analysis and topology*, Historia Mathematica, 35 (2008), pp. 220–241.

[24] S. Mrówka, *Compactness and product spaces*, Colloquium Mathematicae, 7 (1959), pp. 19–22.

[25] nLab authors, *idempotent adjunction*. `https://ncatlab.org/nlab/show/idempotent+adjunction`, July 2024. Revision 23.

[26] ——, *equationally presentable category*. `https://ncatlab.org/nlab/show/equationally+presentable+category`, Feb. 2025. Revision 2.

[27] ——, *Heyting algebra*. `https://ncatlab.org/nlab/show/Heyting+algebra`, Jan. 2025. Revision 61.

[28] ——, *refutation by contradiction*. `https://ncatlab.org/nlab/show/refutation+by+contradiction`, Jan. 2025. Revision 7.

[29] ——, *regular category*. `https://ncatlab.org/nlab/show/regular+category`, Feb. 2025. Revision 79.

[30] J. D. Passmore, *Frames and Ideals in Point-Free Topology*, November 2022. Available at `www.researchgate.net`.

[31] J. Picado and A. Pultr, *Frames and Locales: Topology without points*, Springer Science & Business Media, 2012.

[32] ——, *Notes on Point-Free Topology*, New Perspectives in Algebra, Topology and Categories: Summer School, Louvain-la-Neuve, Belgium, September 12-15, 2018 and September 11-14, 2019, (2021), pp. 173–223.

[33] ——, *Separation in Point-Free Topology*, Springer, 2021.

[34] A. Pultr and J. Sichler, *Frames: Topology Without Points*, in Lattice Theory: Special Topics and Applications: Volume 1, Springer, 2014, pp. 55–88.

[35] A. Pultr and A. Tozzi, *Notes on Kuratowski-Mrówka theorems in point-free context*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 33 (1992), pp. 3–14.

[36] A. Razafindrakoto, *Monadic aspects of the ideal lattice functor on the category of distributive lattices*, arXiv preprint arXiv:2404.19642, (2024).

[37] M. H. Stone, *The theory of representation for boolean algebras*, Transactions of the American Mathematical Society, 40 (1936), pp. 37–111.

[38] ——, *Applications of the theory of boolean rings to general topology*, Transactions of the American Mathematical Society, 41 (1937), pp. 375–481.

[39] A. L. Suarez, *Revisiting the relation between subspaces and sublocales*, 2020.

[40] J. van Oosten, *Basic Category Theory*, Citeseer, 1995.

[41] J. J. C. Vermeulen, *Proper maps of locales*, Journal of Pure and Applied Algebra, 92 (1994), pp. 79–107.

[42] ——, *Some constructive results related to compactness and the (strong) Hausdorff property for locales*, in Category Theory: Proceedings of the International Conference held in Como, Italy, July 22–28, 1990, Springer, 2006, pp. 401–409.