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Frames and Ideals in Point-Free Topology

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Abstract

This report investigates and explores some of the introductory notions in point-free topology. Point-free topology uses the language of category theory and thus we briefly provide the necessary concepts that are needed from category theory for the rest of the report. The central objects of investigation are called *frames*. Frames are complete lattices which satisfy the join-infinite distribution law. The definition of a frame is motivated from the open set lattices of topological spaces. Frames essentially mimic the behaviour of the open sets in a topological space. The report proceeds to uncover the relations between spaces and frames, the functor $\Omega : \mathsf{Top} \longrightarrow \mathsf{Frm}$ is the first example of how spaces are related to frames. We define sober spaces and explain their importance in point-free topology via the full and faithful functor $\Omega : \mathsf{Sob} \longrightarrow \mathsf{Frm}$, in this way the category of sober spaces is embedded into the category of frames and thus we may regard point-free topology as a natural extension of classical topology. Due to the contravariance of Ω, we introduce the category Loc of locales as $\mathsf{Loc} := \mathsf{Frm}^{\mathrm{op}}$, so that $\Omega : \mathsf{Top} \longrightarrow \mathsf{Loc}$ is covariant. We may recover a topological space from a frame via the contravariant functor $\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Top}$. Thus working covariantly, we have a pair of functors $\mathsf{Top} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathsf{Loc}$. We show that Σ is in fact a right adjoint to Ω. Spatiality of frames is discussed and a lattice-theoretic characterisation for spatial frames is presented. Finally we investigate the ideals in frames, the collection of all ideals in a frame constitutes a frame (called the *ideal lattice*). A natural problem to then investigate is the spatiality of the ideal lattice of a frame. Specifically we provide a sufficient condition, while using the Axiom of Choice, for the ideal lattice of a frame to be spatial.

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Contents

1 Introduction

The subject of study in this report is point-free topology. Point-free topology is essentially where classical topology meets lattice theory. We can determine a lot of information about a topological space from its lattice of open sets, for example the notions of compactness; connectedness and continuity depend only on open sets and do not ever mention points [6]. Quite simply, point-free topology is the study of topology where the lattices of open sets are taken as fundamental ([4], p. 41). The objects of study are called frames (or locales) and these are generalised open set lattices.

An appealing aspect of point-free topology is that there are many important results that do not require any form of the Axiom of Choice, whereas their classical counterparts do require choice principles ([7], Preface). A prime example is the point-free Tychonoff theorem: one does not need choice principles to prove that a product of compact locales is compact ([3], p. 86).

By taking the point-free approach, we wish to forget about points in spaces; indeed this is an intuitive and natural way to to view geometry. In classical geometry, objects such as lines and planes are viewed as primitive entities and not as collections of points – the geometry is based on the relations between these objects ([9], p. 55). General (classical) topology takes an analytic approach to geometry, we begin with a set X of points and a structure is determined on X by specifying open subsets ([9], p. 55). Let us take the hint from classical geometry and view our space X similarly, we wish to view the open sets in X as primitive entities and consider the points inside these open sets as other non-primitive entities which are "incident" with the open sets ([7], p. 1). We wish to develop topology starting with the concept of "place" and a natural structure on this system of places ([9], p. 55). These places should be able to be joined to form larger ones, and we should have a way of determining whether two places meet or not – this will naturally invoke lattice theory ([8], p. 175). Indeed, we can already associate points in a space with places; we can view a point as its collection of open neighbourhoods, which is an approximation of the point by places of diminishing size ([7], p. 1). This is where our intuition for point-free topology begins.

2 Concepts Needed from Category Theory

We follow the approach to category theory as is given by Leinster [5] – this will be our central reference for category theory.

We outline only the necessary concepts from category theory which are needed in our study of point-free topology. In point-free topology, our objects of study are *frames*, however we also study their structure preserving maps which are called *frame homomorphisms* – it is therefore natural to group these two concepts together into one central subject of study known as the category Frm. In this investigation of the theory of frames, the language of category theory is used as a means to an end, hence we do not provide in-depth motivation for the defined concepts. Rather we will simply define these tools and concepts, and then exploit them later in the theory of frames.

The idea of category theory is to collect together a system of mathematical objects such that there is a notion of morphism between objects which relates them to one another ([5], p. 9).

2.1 Categories

A category $\mathcal{C}$ is a mathematical structure consisting of a collection of objects and a collection of arrows (morphisms) between objects such that these arrows can be composed associatively. Formally, we define a category as follows.

Definition 2.1.1 (Category)**.** A category $\mathcal{C}$ consists of the following data:

i. A collection of **objects**, denoted by $\mathrm{ob}(\mathcal{C})$.

ii. For each $A, B \in \mathrm{ob}(\mathcal{C})$, a collection $\hom_{\mathcal{C}}(A, B)$ of **morphisms** from A to B. We write $f : A \longrightarrow B$ to mean $f \in \hom_{\mathcal{C}}(A, B)$, we call A the **domain** and B the **codomain** of f.

iii. For each $A, B, C \in \mathrm{ob}(\mathcal{C})$, a function

$$\circ : \hom_{\mathcal{C}}(B, C) \times \hom_{\mathcal{C}}(A, B) \longrightarrow \hom_{\mathcal{C}}(A, C), \text{ where } (g, f) \longmapsto g \circ f,$$

called **composition** of morphisms (we often write gf instead of $g \circ f$) which satisfies the associativity law:

– for all $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$ in $\mathcal{C}$, we have $(hg)f = h(gf)$.

iv. For each $A \in \mathrm{ob}(\mathcal{C})$, a morphism $1_A : A \longrightarrow A$, called the **identity** on A which satisfies:

– for each $f : A \longrightarrow B$ in $\mathcal{C}$, we have $f = f1_A = 1_B f$.

Note that we sometimes express a morphism $f : A \longrightarrow B$ as $A \xrightarrow{f} B$, this helps us streamline our notation, and we often refer to a morphism as an *arrow*. Some common categories are:

- The category Set whose objects are sets and whose morphisms are functions (in the usual sense).
- The category Grp whose objects are groups and whose morphisms are group homomorphisms.
- The category Top whose objects are topological spaces and whose morphisms are continuous functions. This category will be important to us later.

A special type of morphism is that of an *isomorphism* as defined below.

Definition 2.1.2 (Isomorphism)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{ob}(\mathcal{C})$. A morphism $f : A \longrightarrow B$ is called an **isomorphism** iff there exits a morphism $f^{-1} : B \longrightarrow A$ such that $ff^{-1} = 1_B$ and $f^{-1}f = 1_A$. When f^{-1} exists it can be shown that it is unique and hence we call f^{-1} the **inverse** of f. When there is an isomorphism $f : A \longrightarrow B$, we say A and B are **isomorphic** and write $A \cong B$.

2.2 Commutative Diagrams

Within a category, we can often illustrate some of the objects, and the arrows between them, by means of commutative diagrams. For example, let us consider the following interaction played out in some category $\mathcal{C}$:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & D \arrow[dr, "i"] & \\
B \arrow[r, "g"'] & C \arrow[r, "j"'] & E
\end{tikzcd}$$

Here we have illustrated some objects, and the arrows between them, with a diagram. If we have $jgf = ih$, then we say the diagram *commutes*. In general, a diagram is said to commute provided that whenever there are two paths between an object A and an object B, the morphism from A to B obtained by composing along one path is equal to the morphism obtained by composing along the other ([5], p. 11).

2.3 Dual Categories and the Duality Principle

To every category $\mathcal{C}$ there is an associated category $\mathcal{C}^{\mathrm{op}}$ called the *dual* of $\mathcal{C}$, which is simply obtained by 'reversing' each of the arrows in $\mathcal{C}$.

Definition 2.3.1 (Dual category)**.** Let $\mathcal{C}$ be a category. Then the dual category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is itself a category such that:

i. $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$.

ii. For all $A, B \in \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$, $\mathrm{hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{hom}_{\mathcal{C}}(B, A)$. That is, the morphisms $B \xrightarrow{f} A$ in $\mathcal{C}$ are in a one-to-one correspondence with the morphisms $B \xleftarrow{f^{\mathrm{op}}} A$ in $\mathcal{C}^{\mathrm{op}}$. Composition of morphisms in $\mathcal{C}^{\mathrm{op}}$ is defined as follows:

   – If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathcal{C}$ (so that $A \xrightarrow{gf} C$ is their composition), then $A \xleftarrow{f^{\mathrm{op}}} B \xleftarrow{g^{\mathrm{op}}} C$ are morphisms in $\mathcal{C}^{\mathrm{op}}$ with composition $C \xrightarrow{f^{\mathrm{op}} g^{\mathrm{op}}} A$.

iii. For all $A \in \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$, we have $1_A^{\mathrm{op}} = 1_A$ (i.e. the identities in $\mathcal{C}^{\mathrm{op}}$ are precisely the same as the identities in $\mathcal{C}$).

Fundamental to category theory is the *duality principle* which states that every categorical definition, theorem and proof has a *dual*, which is simply obtained by reversing all the arrows within the given statement ([5], p. 16). This is extremely useful, as any true statement in category theory has a dual statement which is also true; this can be a good time-saver when proving theorems as we often get two theorems for the price of one!

2.4 Subcategories

As is usual with most mathematical structures, we can define substructures; in this case we define the notion of a *subcategory*.

Definition 2.4.1 (Subcategory)**.** Let $\mathcal{C}$ be a category. A **subcategory** $\mathcal{D}$ of $\mathcal{C}$ is itself a category such that:

i. $\mathrm{ob}(\mathcal{D}) \subseteq \mathrm{ob}(\mathcal{C})$.

ii. $(\forall\ A, B \in \mathrm{ob}(\mathcal{D}))$ we have $\mathrm{hom}_{\mathcal{D}}(A, B) \subseteq \mathrm{hom}_{\mathcal{C}}(A, B)$, with $\mathrm{hom}_{\mathcal{D}}(A, B)$ closed under composition and identities.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called **full** iff $(\forall\ A, B \in \mathrm{ob}(D))$ $\mathrm{hom}_{\mathcal{D}}(A, B) = \mathrm{hom}_{\mathcal{C}}(A, B)$.

2.5 Functors

Functors provide a notion of structure preserving map between categories.

Definition 2.5.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the following data:

i. A function $\mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$, where $A \longmapsto F(A)$.

ii. For all $A, B \in \mathrm{ob}(\mathcal{C})$, a function $\hom_{\mathcal{C}}(A, B) \longrightarrow \hom_{\mathcal{D}}(F(A), F(B))$, where $\left(A \xrightarrow{f} B\right) \longmapsto \left(F(A) \xrightarrow{F(f)} F(B)\right)$, such that

   – for any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, we have $F(gf) = F(g)F(f)$;
   – for any $A \in \mathrm{ob}(\mathcal{C})$, we have $F(1_A) = 1_{F(A)}$.

We note that a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is often denoted by $\mathcal{C} \xrightarrow{F} \mathcal{D}$, and is sometimes called a *covariant* functor from $\mathcal{C}$ to $\mathcal{D}$. The important property of a functor is that it respects morphism composition. Functors themselves may be composed associatively too. Also, for any category $\mathcal{C}$, we denote by $1_{\mathcal{C}}$ the identity functor on $\mathcal{C}$, which sends every object to itself and every morphism to itself. We therefore may take the perspective of viewing categories as objects and functors as arrows between categories. This leads us to the category CAT, whose objects are categories and whose arrows are functors.

The notion of *contravariant* functors will play an important role in our study of frames, we define the concept here.

Definition 2.5.2 (Contravariant functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **contravariant** functor F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data ([7], p. 340):

i. A function $\mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$, where $A \longmapsto F(A)$.

ii. For all $A, B \in \mathrm{ob}(\mathcal{C})$, a function $\hom_{\mathcal{C}}(A, B) \longrightarrow \hom_{\mathcal{D}}(F(B), F(A))$, where $\left(A \xrightarrow{f} B\right) \longmapsto \left(F(B) \xrightarrow{F(f)} F(A)\right)$, such that

   – for any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, we have $F(gf) = F(f)F(g)$;
   – for any $A \in \mathrm{ob}(\mathcal{C})$, we have $F(1_A) = 1_{F(A)}$.

Definition 2.5.3 (Faithful and full functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. Then:

i. F is **faithful** iff, for all $A, B \in \mathrm{ob}(\mathcal{C})$, the function $\hom_{\mathcal{C}}(A, B) \longrightarrow \hom_{\mathcal{D}}(F(A), F(B))$ is injective.

ii. F is **full** iff, for all $A, B \in \mathrm{ob}(\mathcal{C})$, the function $\hom_{\mathcal{C}}(A, B) \longrightarrow \hom_{\mathcal{D}}(F(A), F(B))$ is surjective.

iii. F is an **embedding** iff F is faithful and full.

We define faithfulness and fullness for contravariant functors similarly.

2.5.1 An Observation

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ may be viewed as a covariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}^{\mathrm{op}}$. To see this, let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in $\mathcal{C}$. Then we have $A \xrightarrow{gf} C$ in $\mathcal{C}$ and $F(C) \xrightarrow{F(gf)} F(A)$ in $\mathcal{D}$. Furthermore, $F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$ are morphisms in $\mathcal{D}$ with the composition $F(C) \xrightarrow{F(f)F(g)} F(A)$. By contravariance, we have $F(gf) = F(f)F(g)$. Therefore, $F(gf)^{\mathrm{op}} = (F(f)F(g))^{\mathrm{op}}$ in $\mathcal{D}^{\mathrm{op}}$. Now consider the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd} F(C) \arrow[r, "F(g)"] \arrow[dr, "F(f)F(g)"'] & F(B) \arrow[d, "F(f)"] \\ & F(A) \end{tikzcd}$$

By reversing each of the arrows, we obtain a commutative diagram in $\mathcal{D}^{\mathrm{op}}$:

$$\begin{tikzcd} F(C) & F(B) \arrow[l, "F(g)^{\mathrm{op}}"'] \\ & F(A) \arrow[u, "F(f)^{\mathrm{op}}"'] \arrow[ul, "(F(f)F(g))^{\mathrm{op}}"] \end{tikzcd}$$

In order for this second diagram to commute, we must have $(F(f)F(g))^{\mathrm{op}} = F(g)^{\mathrm{op}}F(f)^{\mathrm{op}}$. Therefore, $F(gf)^{\mathrm{op}} = F(g)^{\mathrm{op}}F(f)^{\mathrm{op}}$ and we may conclude that $F : \mathcal{C} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is covariant. In a similar manner, it can be shown that a contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ may be viewed as a covariant functor $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$.

2.6 Natural Transformations

Natural transformations are the structure preserving maps between functors.

Definition 2.6.1 (Natural transformation)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Consider functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$. A **natural transformation** α from F to G (denoted $F \overset{\alpha}{\Rightarrow} G$) is a collection of arrows in $\mathcal{D}$, $\alpha := \{F(A) \xrightarrow{\alpha_A} G(A) : A \in \mathrm{ob}(\mathcal{C})\}$, such that, for every arrow $A \xrightarrow{f} B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

The α_A are known as the **components** of the natural transformation α.

2.7 Adjunctions

The final concept that we need from category theory is that of *adjoint functors*. This notion will prove to be useful in our study of the theory of frames.

Definition 2.7.1 (Adjoint functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Consider functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$. Let η and ε be natural transformations such that $1_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ and $FG \overset{\varepsilon}{\Rightarrow} 1_{\mathcal{D}}$. We say that

F is **left adjoint** to G, and that G is **right adjoint** to F, iff, for all $A \in \mathrm{ob}(\mathcal{C})$ and $B \in \mathrm{ob}(\mathcal{D})$, the diagrams below commute:

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[d, "1_{F(A)}"'] & FGF(A) \arrow[dl, "\varepsilon_{F(A)}"] \\ F(A) & \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[d, "1_{G(B)}"'] & GFG(B) \arrow[dl, "G(\varepsilon_B)"] \\ G(B) & \end{tikzcd}$$

When $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$ are adjoint functors, with F left adjoint to G (and hence G right adjoint to F), we write $F \dashv G$, and say that there is an **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ (η is known as the **unit** and ε the **counit** of the adjunction).

3 Open Set Lattices and Frames

When we study topological spaces from the classical perspective, we first start with a set X of points and then we consider a family τ of open sets which defines a topology on X. In this classical setting, the points themselves are needed to define the open sets (the points sit inside the open sets). In the point-free setting, we wish to forget about points entirely and purely study the open sets without reference to the points inside them. This requires us to view the open sets as primitive entities [4] in their own right and investigate their relations between each other. The place to start is by considering the inclusion ordering relation on the topology τ. We will then define a frame as a point-free generalisation of a topology. This point-free approach naturally invokes the use of lattice theory.

3.1 Open Set Lattices

Definition 3.1.1. Let X be a space with topology τ. We define $\Omega(X)$ to be the poset $(\tau, \subseteq)$.

For a space X, the poset $\Omega(X)$ has a bottom $\emptyset$ and a top X. Moreover, since an arbitrary union of open sets is an open set and a finite intersection of open sets is an open set, it follows that $\Omega(X)$ is a *lattice* in which, for any $U, V \in \Omega(X)$, $U \vee V = U \cup V$ and $U \wedge V = U \cap V$. $\Omega(X)$ is in fact a *distributive* lattice since the set-theoretic operations of union and intersection distribute over each other.

Proposition 3.1.2. *Let X be a space. Then $\Omega(X)$ is a complete lattice.*

Proof. It is sufficient for us to show that $\Omega(X)$ is closed under arbitrary joins. For any $\{U_i\}_{i\in I} \subseteq \Omega(X)$, we have $\bigcup_{i\in I} U_i \in \Omega(X)$ and thus $\Omega(X) \ni \bigvee_{i\in I} U_i \coloneqq \bigcup_{i\in I} U_i$. $\square$

We will refer to $\Omega(X)$ as the *open set lattice* of the space X. We need to be careful when taking infinite meets in $\Omega(X)$; an infinite intersection of open sets is in general not an open set and thus (in general) $\bigwedge_{i\in I} U_i \neq \bigcap_{i\in I} U_i$. However, finite meet *is* given by finite intersection.

Proposition 3.1.3. *Let X be a space and consider its open set lattice $\Omega(X)$. Then, for all $\{U_i\}_{i\in I} \subseteq \Omega(X)$, we have $\bigwedge_{i\in I} U_i = \mathrm{int}(\bigcap_{i\in I} U_i)$.*

Proof. Let $\{U_i\}_{i\in I} \subseteq \Omega(X)$. Now $\mathrm{int}(\bigcap_{i\in I} U_i)$ is the largest open set contained in $\bigcap_{i\in I} U_i$ and thus $\mathrm{int}(\bigcap_{i\in I} U_i) \in \Omega(X)$. Since $\bigcap_{i\in I} U_i \subseteq U_j$ for each $j \in I$, we have $\mathrm{int}(\bigcap_{i\in I} U_i) \subseteq U_j$ for each $j \in I$. Thus $\mathrm{int}(\bigcap_{i\in I} U_i)$ is a lower bound of $\{U_i\}_{i\in I}$ in $\Omega(X)$. Let $A \in \Omega(X)$ be a lower bound of $\{U_i\}_{i\in I}$ in $\Omega(X)$. Then $A \subseteq U_j$ for each $j \in I$ and hence $A \subseteq \bigcap_{i\in I} U_i$. As $\mathrm{int}(\bigcap_{i\in I} U_i)$ is the largest open set contained in $\bigcap_{i\in I} U_i$, it follows that $A \subseteq \mathrm{int}(\bigcap_{i\in I} U_i)$ and hence $\mathrm{int}(\bigcap_{i\in I} U_i) = \bigwedge_{i\in I} U_i$. $\square$

Proposition 3.1.4. *Let X be a space and consider its open set lattice $\Omega(X)$. Then, for all $\{U_i\}_{i\in I} \subseteq \Omega(X)$ and all $V \in \Omega(X)$, we have*

$$V \wedge \bigvee_{i\in I} U_i = \bigvee_{i\in I} V \wedge U_i. \tag{3.1}$$

Proof. Let $\{U_i\}_{i\in I} \subseteq \Omega(X)$ and $V \in \Omega(X)$. Then:

$$\begin{aligned} V \wedge \bigvee_{i\in I} U_i &= V \cap \bigcup_{i\in I} U_i \\ &= \bigcup_{i\in I} V \cap U_i \\ &= \bigvee_{i\in I} V \wedge U_i. \end{aligned}$$

□

Given a space X, with $A \in \Omega(X)$ and $\{B_i\}_{i\in I} \subseteq \Omega(X)$, we have (in general)

$$A \vee \bigwedge_{i\in I} B_i \neq \bigwedge_{i\in I} A \vee B_i.$$

However, for a *finite* subset $\mathcal{B} = \{B_1, \ldots, B_k\} \subseteq \Omega(X)$ we do have $A \vee (\bigwedge \mathcal{B}) = \bigwedge\{A \vee B : B \in \mathcal{B}\}$ (this is because finite meet is given by finite intersection).

Example 3.1.5. Take the real line $\mathbb{R}$ with its usual topology. We consider the intervals $A = (1,2)$ and, for each $n \in \mathbb{N}$, $B_n = (-1/n, 1+1/n)$, so that $\bigcap_{n\in\mathbb{N}} B_n = [0,1]$. We have $A \vee \bigwedge_{n\in\mathbb{N}} B_n = A \cup \mathrm{int}(\bigcap_{n\in\mathbb{N}} B_n) = (1,2) \cup (0,1)$. Now $\bigwedge_{n\in\mathbb{N}} A \vee B_n = \mathrm{int}(\bigcap_{n\in\mathbb{N}} A \cup B_n) = \mathrm{int}(A \cup \bigcap_{n\in\mathbb{N}} B_n) = \mathrm{int}([0,2)) = (0,2)$. Hence $A \vee \bigwedge_{n\in\mathbb{N}} B_n \neq \bigwedge_{n\in\mathbb{N}} A \vee B_n$.

3.2 Frames

Opting for a point-free approach to topology, we should head towards the direction of generalising the notion of 'open set lattice'; in particular we do not wish to make reference to points. Such open set lattices have two fundamental properties: firstly, open set lattices are complete lattices and, secondly, they satisfy the distribution law (3.1). These two properties are what we will use to define our point-free generalisations of open set lattices.

Definition 3.2.1 (Join-infinite distributivity)**.** A complete lattice L is said to satisfy the **join-infinite distributivity** law ([2], p. 240) iff

$$(\forall \{x_i\}_{i\in I} \subseteq L)\ (\forall y \in L)\ \text{ we have } y \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I} y \wedge x_i.$$

Lemma 3.2.2. *Let L be a complete lattice, let $y \in L$ and $\{x_i\}_{i\in I} \subseteq L$. Then*

$$\bigvee_{i\in I} y \wedge x_i \leqslant y \wedge \bigvee_{i\in I} x_i.$$

Proof. We have, for each $i \in I$, $y \wedge x_i \leqslant x_i \leqslant \bigvee_{i\in I} x_i$. Hence $y \wedge (y \wedge x_i) = y \wedge x_i \leqslant y \wedge \bigvee_{i\in I} x_i$. So $y \wedge \bigvee_{i\in I} x_i$ is an upper bound of the set $\{y \wedge x_i : i \in I\}$. Thus $\bigvee_{i\in I} y \wedge x_i \leqslant y \wedge \bigvee_{i\in I} x_i$. □

Hence, to check whether a complete lattice L satisfies the join-infinite distributivity law, we need only verify that $\bigvee_{i\in I} y \wedge x_i \geqslant y \wedge \bigvee_{i\in I} x_i$ for all $\{x_i\}_{i\in I} \subseteq L$ and $y \in L$.

Definition 3.2.3 (Frame)**.** A **frame** is a complete lattice L which satisfies the join-infinite distributivity law.

Our prototypical model of a frame is the open set lattice $\Omega(X)$ of some space X. However, any finite distributive lattice is also a frame since all joins are finite. Now having defined the concept of a frame, we have at hand a point-free structure which mimics the behaviour of open sets [7]. Quite clearly, we see that a frame is a generalisation of an open set lattice.

We now wish to study frames as their own algebraic structure; we therefore need a notion of *structure preserving map* or *frame homomorphism*. In classical topology, continuous functions play the role of structure preserving maps (continuous functions are the morphisms in classical topology). Let X and Y be spaces. For every continuous map $f : X \longrightarrow Y$, there is an associated map (see [8]) $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$ such that

$$(\forall\, U \in \Omega(Y))\ \ \Omega(f)(U) \coloneqq f^{-1}[U] \in \Omega(X). \tag{3.2}$$

Proposition 3.2.4. *Let X and Y be spaces. Let $f : X \longrightarrow Y$ be a continuous function. Then the map $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$, defined by $\Omega(Y) \ni U \longmapsto f^{-1}[U] \in \Omega(X)$, preserves arbitrary joins and binary meets.*

Proof. Let $\{U_i\}_{i\in I} \subseteq \Omega(Y)$. Then $\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$. We have

$$\begin{aligned}\Omega(f)(\bigcup_{i\in I} U_i) &= f^{-1}[\bigcup_{i\in I} U_i]\\ &= \bigcup_{i\in I} f^{-1}[U_i]\\ &= \bigcup_{i\in I} \Omega(f)(U_i).\end{aligned}$$

Let $A, B \in \Omega(Y)$. Then $A \wedge B = A \cap B$. We have

$$\begin{aligned}\Omega(f)(A \cap B) &= f^{-1}[A \cap B]\\ &= f^{-1}[A] \cap f^{-1}[B]\\ &= \Omega(f)(A) \cap \Omega(f)(B).\end{aligned}$$

□

We note that, for an arbitrary subset $\{U_i\}_{i\in I} \subseteq \Omega(Y)$, we have $\bigwedge_{i\in I} U_i = \operatorname{int}(\bigcap_{i\in I} U_i)$ and thus we expect that (for infinite meets) $\Omega(f)(\bigwedge_{i\in I} U_i) \neq \bigwedge_{i\in I} \Omega(f)(U_i)$ in general. Since the mappings of the form $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$ preserve arbitrary joins and binary meets, we adopt this notion of preservation as our definition for *frame homomorphism* [8].

Definition 3.2.5 (Frame homomorphism)**.** Let L and M be frames. We define a **frame homomorphism** h from L to M as a map $h : L \longrightarrow M$ such that:

i. $(\forall\, \{x_i\}_{i\in I} \subseteq L)\ \ h(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} h(x_i)$;

ii. $(\forall\, a, b \in L)\ \ h(a \wedge b) = h(a) \wedge h(b)$.

Further, a frame homomorphism $h : L \longrightarrow M$ is said to be a frame **isomorphism** iff h is also a bijection. If $h : L \longrightarrow M$ is a frame isomorphism, we say that L and M are frame-isomorphic (or, simply, isomorphic) and write $L \cong M$.

From this point forward, given spaces X,Y and a continuous function $f : X \longrightarrow Y$, we denote by $\Omega(f)$ the frame homomorphism as defined in (3.2).

Proposition 3.2.6. *Let $h : L \longrightarrow M$ be a frame homomorphism. Then $h(0) = 0$ and $h(1) = 1$.*

Proof. Consider the empty subset $\emptyset$ of L. We have $\emptyset^u := \{x \in L : (\forall a \in \emptyset)\ a \leqslant x\} = L$. Therefore $\bigvee \emptyset$ is equal to the least element of L: $\bigvee \emptyset = 0$. Also $\emptyset^\ell := \{x \in L : (\forall a \in \emptyset)\ x \leqslant a\} = L$. Therefore $\bigwedge \emptyset$ is equal to the greatest element of L: $\bigwedge \emptyset = 1$. Now $h(0) = h(\bigvee \emptyset) = \bigvee\{h(x) \in M : x \in \emptyset\} = \bigvee \emptyset = 0 \in M$ and, for each $x, y \in \emptyset$, $h(1) = h(\bigwedge \emptyset) = h(\bigwedge\{x, y\}) = \bigwedge\{h(x), h(y)\} = \bigwedge \emptyset = 1 \in M$. □

Proposition 3.2.7. *The composition of two frame homomorphisms (when defined) is again a frame homomorphism.*

Proof. Let $L \xrightarrow{h} M \xrightarrow{g} N$ be frame homomorphisms and consider their composition $L \xrightarrow{gh} N$. Let $\{x_i\}_{i\in I} \subseteq L$ and $a, b \in L$. Then $gh(\bigvee_{i\in I} x_i) = g(h(\bigvee_{i\in I} x_i)) = g(\bigvee_{i\in I} h(x_i)) = \bigvee_{i\in I} g(h(x_i)) = \bigvee_{i\in I} gh(x_i)$ and $gh(a \wedge b) = g(h(a \wedge b)) = g(h(a) \wedge h(b)) = g(h(a)) \wedge g(h(b)) = gh(a) \wedge gh(b)$. □

Frame homomorphisms can be composed associatively and the identity map $1_L : L \longrightarrow L$ on a frame L is trivially a frame homomorphism. Thus frames and their homomorphisms yield a category, which we call Frm.

Definition 3.2.8 (The category Frm)**.** We denote by Frm the category whose objects are frames and whose morphisms are frame homomorphisms.

3.2.1 An Observation and the Functor Ω

Let us now take note of the following two points:

- Ω maps any topological space X to the frame $\Omega(X)$, $X \longmapsto \Omega(X)$. That is, Ω is a mapping $\mathrm{ob}(\mathsf{Top}) \longrightarrow \mathrm{ob}(\mathsf{Frm})$.
- Ω maps any continuous function $X \xrightarrow{f} Y$ to the frame homomorphism $\Omega(X) \xleftarrow{\Omega(f)} \Omega(Y)$, $\hom_{\mathsf{Top}}(X, Y) \ni f \longmapsto \Omega(f) \in \hom_{\mathsf{Frm}}(\Omega(Y), \Omega(X))$. That is, for all $X, Y \in \mathrm{ob}(\mathsf{Top})$, Ω is a mapping $\hom_{\mathsf{Top}}(X, Y) \longrightarrow \hom_{\mathsf{Frm}}(\Omega(Y), \Omega(X))$.

Proposition 3.2.9. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in Top. Then $\Omega(gf) = \Omega(f)\Omega(g)$.*

Proof. We note that: $\Omega(Z) \xrightarrow{\Omega(g)} \Omega(Y)$; $\Omega(Y) \xrightarrow{\Omega(f)} \Omega(X)$; $\Omega(Z) \xrightarrow{\Omega(gf)} \Omega(X)$ and $\Omega(Z) \xrightarrow{\Omega(f)\Omega(g)} \Omega(X)$. For all $A \in \Omega(Z)$, we have: $\Omega(gf)(A) = (gf)^{-1}[A] = f^{-1}[g^{-1}[A]] = \Omega(f)(\Omega(g)(A)) = (\Omega(f)\Omega(g))(A)$. Thus $\Omega(gf) = \Omega(f)\Omega(g)$. □

Proposition 3.2.10. *Let $X \in \mathrm{ob}(\mathsf{Top})$. Then $\Omega(1_X) = 1_{\Omega(X)}$.*

Proof. For all $U \in \Omega(X)$, $1_{\Omega(X)}(U) = U$ and $\Omega(1_X)(U) = 1_X^{-1}[U] = \{x \in X : 1_X(x) \in U\} = \{x \in X : x \in U\} = U$. Thus $\Omega(1_X) = 1_{\Omega(X)}$. □

We may therefore conclude that we have a *contravariant* functor $\Omega : \mathsf{Top} \longrightarrow \mathsf{Frm}$.

Definition 3.2.11 (The category of locales)**.** We define the category of **locales** (whose objects are frames, here known as **locales**, and whose morphisms are called **localic maps** [7]), denoted by Loc, as $\mathsf{Loc} := \mathsf{Frm}^{\mathrm{op}}$.

With this notation, and in light of 2.5.1, we may view Ω as a covariant functor $\Omega : \mathsf{Top} \longrightarrow \mathsf{Loc}$.

4 Sober Spaces: Motivation for Point-Free Topology

Sober spaces provide us with a large class of topological spaces such that no information is lost about these spaces when studying their open set lattices; we will see that a sober space can be homeomorphically reconstructed from its open set lattice [8]. This provides us with the impetus to move away from classical topology towards the point-free setting where we deliberately study these open set lattices and frames.

4.1 Sobriety

Definition 4.1.1 (Meet-irreducible elements)**.** Let L be a lattice. An element $x \in L$ is said to be **meet-irreducible** (or **prime** [8]) iff:

i. $x \neq 1$ (provided L has a 1);

ii. $(\forall\, a, b \in L) \quad x = a \wedge b \implies x = a$ or $x = b$.

For distributive lattices, the second meet-irreducibility requirement (ii. above) is simpler.

Proposition 4.1.2. *For a distributive lattice L, let $x \in L$ be such that $x \neq 1$ (provided L has a 1). Then x is meet-irreducible iff*

$$(\forall\, a, b \in L) \quad a \wedge b \leqslant x \implies a \leqslant x \text{ or } b \leqslant x.$$

Proof. ($\Rightarrow$) Assume that x is meet-irreducible. Let $a, b \in L$ be such that $a \wedge b \leqslant x$. Then $x = x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b)$. So, $x = x \vee a$ or $x = x \vee b$. Thus $a \leqslant x$ or $b \leqslant x$.
($\Leftarrow$) Assume that $(\forall a, b \in L)\ a \wedge b \leqslant x \implies a \leqslant x$ or $b \leqslant x$. Let $c, d \in L$ be such that $x = c \wedge d$. So $x \leqslant c$ and $x \leqslant d$. But we also have $c \wedge d \leqslant x$ and hence, by assumption, $c \leqslant x$ or $d \leqslant x$. Therefore $x = c$ or $x = d$. □

Let X be a space and consider its open set lattice $\Omega(X)$. For all $x \in X$, $\mathrm{cl}\{x\}$ is the smallest closed set containing $\{x\}$, and thus $(\mathrm{cl}\{x\})^c \coloneqq X \setminus \mathrm{cl}\{x\}$ is an open set. That is to say, $(\mathrm{cl}\{x\})^c \in \Omega(X)$.

Proposition 4.1.3. *In $\Omega(X)$, $(\mathrm{cl}\{x\})^c$ is meet-irreducible for all $x \in X$.*

Proof. Firstly $(\mathrm{cl}\{x\})^c \coloneqq X \setminus \mathrm{cl}\{x\}$ (and $\mathrm{cl}\{x\} \neq \emptyset$), hence $(\mathrm{cl}\{x\})^c \neq X$. Now let $U, V \in \Omega(X)$ and suppose $U \cap V \subseteq (\mathrm{cl}\{x\})^c$. Then $x \notin U \cap V$. This implies $x \notin U$ or $x \notin V$. Suppose that it is $x \notin U$. Then $x \in U^c$, where U^c is a closed set. But $\mathrm{cl}\{x\}$ is the smallest closed set containing x, so $\mathrm{cl}\{x\} \subseteq U^c$, which implies $U \subseteq (\mathrm{cl}\{x\})^c$. Therefore, $(\mathrm{cl}\{x\})^c$ is meet-irreducible in $\Omega(X)$. □

Definition 4.1.4 (Sobriety)**.** A space X is said to be **sober** ([7], p. 2) iff, for all $U \in \Omega(X)$,

$$U \text{ is meet irreducible} \implies (\exists!\, x \in X) \ :\ U = (\mathrm{cl}\{x\})^c.$$

Proposition 4.1.5. *Every sober space is T_0.* (See [10]).

Proof. We will prove the contrapositive. Let X be a space and suppose that X is not T_0. This means there exist *distinct* $x, y \in X$ such that, for all $U \in \Omega(X)$, either $x, y \in U$ or $x, y \notin U$. Now $x \notin (\mathrm{cl}\{x\})^c$, so $y \notin (\mathrm{cl}\{x\})^c$ (since X is not T_0), and similarly $y \notin (\mathrm{cl}\{y\})^c$, so $x \notin (\mathrm{cl}\{y\})^c$. Then we have $y \in \mathrm{cl}\{x\}$ and $x \in \mathrm{cl}\{y\}$, hence $\{y\} \subseteq \mathrm{cl}\{x\}$ and $\{x\} \subseteq \mathrm{cl}\{y\}$. By the properties of closure operators, we have $\mathrm{cl}\{y\} \subseteq \mathrm{cl}\{x\}$ and $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\{y\}$. Hence $\mathrm{cl}\{x\} = \mathrm{cl}\{y\}$, so $(\mathrm{cl}\{x\})^c = (\mathrm{cl}\{y\})^c$. Therefore, we have a meet-irreducible element $A \in \Omega(X)$ such that $A = (\mathrm{cl}\{x\})^c = (\mathrm{cl}\{y\})^c$ for *distinct* points $x, y \in X$. Therefore, X is not sober. □

Sobriety is not a rare property of topological spaces [7, 9], in fact, every T_2 space is sober.

Proposition 4.1.6. *Every T_2 space is sober.*

Proof. Let X be a T_2 space. Let $U \in \Omega(X)$ be such that, for all $x \in X$, $U \neq (\mathrm{cl}\{x\})^c$. Assume that U is meet-irreducible. Let $x, y \in X$ be distinct points such that $x, y \notin U$. Since X is T_2, there exists $U_1, U_2 \in \Omega(X)$ with $U_1 \cap U_2 = \emptyset$ such that $x \in U_1$ and $y \in U_2$. Then

$$U \subsetneq U \cup U_1 \text{ and } U \subsetneq U \cup U_2. \tag{4.1}$$

Now consider the intersection $(U \cup U_1) \cap (U \cup U_2) = [(U \cup U_1) \cap U] \cup [(U \cup U_1) \cap U_2] = U \cup (U \cap U_2) = U$. Since U is meet-irreducible, this implies $U = U \cup U_1$ or $U = U \cup U_2$. But this contradicts (4.1) above. Therefore it must be the case that U is not meet-irreducible. It follows that the only meet-irreducible elements in $\Omega(X)$ are those of the form $(\mathrm{cl}\{a\})^c$, $a \in X$, and thus X is sober. $\square$

The fact that every T_2 space is sober provides us with a large class of interesting sober topological spaces (just think of the metric spaces – every metric space is T_2 and thus sober), this is sufficient motivation for us to pursue sober spaces further.

Definition 4.1.7 (Filters)**.** Let L be a lattice and F a non-empty subset of L. Then F is said to be a **filter** iff:

i. $(\forall x, y \in L)\ x, y \in F \implies x \wedge y \in F$;

ii. $(\forall x \in L)(\forall y \in F)\ y \leqslant x \implies x \in F$.

We can think of a filter F as a non-empty up-set in L that is closed under binary meet. Also, a filter F is called *proper* iff $F \neq L$.

Definition 4.1.8 (Prime filters)**.** A proper filter F in a lattice L is said to be **prime** iff $(\forall a, b \in L)\ a \vee b \in F \implies a \in F$ or $b \in F$.

Definition 4.1.9 (Completely prime filters)**.** A proper filter F in a complete lattice L is said to be **completely prime** iff, for any $\{x_i\}_{i \in I} \subseteq L$, we have

$$\bigvee_{i \in I} x_i \in F \implies (\exists i \in I)\ x_i \in F.$$

When F is a completely prime filter in a complete lattice L, we will refer to F as a *cp-filter* (note that for a filter F to be completely prime it is necessary that F be proper). We will now show that the family of open neighbourhoods of any point in a space X is a cp-filter in $\Omega(X)$. Note that for a space X we will use the notation $\mathcal{U}(x) \coloneqq \{U \in \Omega(X) : x \in U\}$.

Proposition 4.1.10. *Let X be a space and consider $x \in X$. Then $\mathcal{U}(x)$ is a cp-filter in $\Omega(X)$.*

Proof. We first show that $\mathcal{U}(x)$ is a proper filter in $\Omega(X)$. Clearly $\mathcal{U}(x) \subseteq \Omega(X)$. Note that $\mathcal{U}(x)$ is non-empty because $X \in \mathcal{U}(x)$. Let $U, V \in \mathcal{U}(x)$, then $x \in U$ and $x \in V$, with U and V both open. Hence $x \in U \cap V$ and $U \cap V$ is open. Thus $U \cap V \in \mathcal{U}(x)$. Let $W \in \Omega(X)$ be such that $U \subseteq W$. Then $x \in W$, so $W \in \mathcal{U}(x)$. Therefore, $\mathcal{U}(x)$ is a filter in $\Omega(X)$. Note that $\mathcal{U}(x)$ is a *proper* a filter because $\emptyset \notin \mathcal{U}(x)$. Next we show that $\mathcal{U}(x)$ is completely prime. Let $\{A_i\}_{i \in I} \subseteq \Omega(X)$, then we have that $\bigcup_{i \in I} A_i \in \Omega(X)$. Assume that $\bigcup_{i \in I} A_i \in \mathcal{U}(x)$. Then $x \in \bigcup_{i \in I} A_i$. Hence, there exists an $i \in I$ such that $x \in A_i$, and so $A_i \in \mathcal{U}(x)$. We may conclude that the proper filter $\mathcal{U}(x) \subsetneq \Omega(X)$ is completely prime. $\square$

We now present a complete characterisation for sobriety ([9], p. 57).

Lemma 4.1.11. *Let X be a space. Then X is sober iff each cp-filter in $\Omega(X)$ is of the form $\mathcal{U}(x)$ for some $x \in X$.*

Proof. ($\Rightarrow$) Assume that X is sober and let $\mathcal{F} \subsetneq \Omega(X)$ be a cp-filter. Put $V_0 := \bigcup\{V \in \Omega(X) : V \notin \mathcal{F}\} \in \Omega(X)$. Since $\mathcal{F}$ is completely prime, we have, for any $\{U_i\}_{i\in I} \subseteq \Omega(X)$,

$$\Big(\bigcup_{i\in I} U_i \in \mathcal{F} \implies \exists i \in I,\ U_i \in \mathcal{F}\Big) \iff \Big(\forall i \in I,\ U_i \notin \mathcal{F} \implies \bigcup_{i\in I} U_i \notin \mathcal{F}\Big).$$

It thus follows that $V_0 \notin \mathcal{F}$. Also, since $\mathcal{F}$ is a filter, $X \in \mathcal{F}$ and hence $V_0 \neq X$. By definition of V_0 and the fact that $V_0 \notin \mathcal{F}$, it follows that V_0 is the largest open set that is not an element of $\mathcal{F}$. For all $U \in \Omega(X)$, $U \in \mathcal{F} \iff U \notin \{V \in \Omega(X) : V \notin \mathcal{F}\} \iff U \nsubseteq \bigcup\{V \in \Omega(X) : V \notin \mathcal{F}\} \iff U \nsubseteq V_0$. Let $U_1, U_2 \in \Omega(X)$ and suppose $U_1 \cap U_2 \subseteq V_0$. Then $U_1 \cap U_2 \notin \mathcal{F}$. Since $\mathcal{F}$ is a filter,

$$(U_1, U_2 \in \mathcal{F} \implies U_1 \cap U_2 \in \mathcal{F}) \iff (U_1 \cap U_2 \notin \mathcal{F} \implies U_1 \notin \mathcal{F} \text{ or } U_2 \notin \mathcal{F}).$$

Without loss of generality, suppose that $U_1 \notin \mathcal{F}$. Then $U_1 \subseteq V_0$. Therefore, we have obtained the implication

$$U_1 \cap U_2 \subseteq V_0 \implies U_1 \subseteq V_0,$$

and it follows that V_0 is meet-irreducible. But X is sober: there exists a unique $x_0 \in X$ such that $V_0 = (\mathrm{cl}\{x_0\})^c$. Hence, $(\forall\, U \in \Omega(X))\ U \in \mathcal{F} \iff U \nsubseteq (\mathrm{cl}\{x_0\})^c \iff x_0 \in U \iff U \in \mathcal{U}(x_0)$. Therefore, $\mathcal{F} = \mathcal{U}(x_0)$.

($\Leftarrow$) Assume that each cp-filter in $\Omega(X)$ is of the form $\mathcal{U}(x)$ for some $x \in X$. Let $V_0 \in \Omega(X)$ be meet-irreducible. Put $\mathcal{F} := \{U \in \Omega(X) : U \nsubseteq V_0\}$ (note that $\mathcal{F}$ is non-empty because $X \in \mathcal{F}$), we will show that $\mathcal{F}$ is a cp-filter. Let $U_1, U_2 \in \mathcal{F}$. Then $U_1, U_2 \nsubseteq V_0$, so $U_1 \cap U_2 \nsubseteq V_0$ (by the contrapositive of meet-irreducibility). Hence, $U_1 \cap U_2 \in \mathcal{F}$. Let $A \in \Omega(X)$ be such that $U_1 \subseteq A$. Since $U_1 \nsubseteq V_0$, it follows that $A \nsubseteq V_0$ and hence $A \in \mathcal{F}$. Therefore $\mathcal{F}$ is a filter (note that it is *proper* because $V_0 \notin \mathcal{F}$). Let $\{U_i\}_{i\in I} \subseteq \Omega(X)$ and assume $\bigcup_{i\in I} U_i \in \mathcal{F}$. Then $\bigcup_{i\in I} U_i \nsubseteq V_0$, which implies $(\exists i \in I)\ U_i \nsubseteq V_0$ and hence $(\exists i \in I)\ U_i \in \mathcal{F}$. Therefore $\mathcal{F}$ is a cp-filter. By assumption, there exists $x_0 \in X$ such that $\mathcal{F} = \mathcal{U}(x_0)$. Then $(\forall\, U \in \Omega(X))$:

$$\begin{aligned}
&(U \nsubseteq V_0 \text{ iff } x_0 \in U)\\
\iff &(U \subseteq V_0 \text{ iff } x_0 \notin U \text{ iff } x_0 \in U^c \text{ iff } \mathrm{cl}\{x_0\} \subseteq U^c)\\
\iff &(U \subseteq V_0 \text{ iff } U \subseteq (\mathrm{cl}\{x_0\})^c)\\
\iff &V_0 = (\mathrm{cl}\{x_0\})^c.
\end{aligned}$$

Therefore X is sober. □

4.2 Embedding Sober Spaces into Frames

We have introduced sober spaces but, as of yet, we have not seen their usefulness in point-free topology. We will now present an important theorem which implies that the category of sober spaces is embedded into the category of frames under the functor Ω. But first we need some preliminary lemmas.

Lemma 4.2.1. *Let X and Y be spaces such that Y is T_0. Let $f, g : X \longrightarrow Y$ be distinct continuous functions. Then $\Omega(f) \neq \Omega(g)$.*

Proof. Let $x \in X$ be such that $f(x) \neq g(x)$. Let $U \in \Omega(Y)$ be such that $f(x) \in U$ and $g(x) \notin U$ (which is possible since Y is T_0). Then $x \in f^{-1}[U]$ and $x \notin g^{-1}[U]$, i.e. $x \in \Omega(f)(U)$ and $x \notin \Omega(g)(U)$. So $\Omega(f)(U) \neq \Omega(g)(U)$ and thus $\Omega(f) \neq \Omega(g)$. $\square$

Lemma 4.2.2. *Let $h : L \longrightarrow M$ be a frame homomorphism and let $F \subsetneq M$ be a cp-filter. Then $h^{-1}[F]$ is a cp-filter.*

Proof. Since F is a filter, we have $1 \in F$. So $1 = h(1) \in F \iff 1 \in h^{-1}[F]$. Hence $h^{-1}[F]$ is non-empty. Since F is a proper filter, we have $0 \notin F$. So $0 = h(0) \notin F \iff 0 \notin h^{-1}[F]$. Hence $h^{-1}[F]$ is a proper subset of L. Let $a, b \in h^{-1}[F]$. Then $h(a), h(b) \in F$ and $h(a) \wedge h(b) = h(a \wedge b) \in F$, so $a \wedge b \in h^{-1}[F]$. Suppose $c \in L$ is such that $a \leqslant c$. Then $a = a \wedge c$, so $h(a) = h(a \wedge c) = h(a) \wedge h(c) \iff h(a) \leqslant h(c)$ where $h(a) \in F$ and $h(c) \in M$. But F is a filter, so $h(c) \in F$, which means that $c \in h^{-1}[F]$. Finally let $\{x_i\}_{i \in I} \subseteq L$ be such that $\bigvee_{i \in I} x_i \in h^{-1}[F]$. Then $h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} h(x_i) \in F$, which implies that $h(x_i) \in F$ for some $i \in I$ (F is completely prime). But then $x_i \in h^{-1}[F]$. We may thus conclude that $h^{-1}[F]$ is a cp-filter. $\square$

Theorem 4.2.3. *Let X be a space and let Y be a sober space. Then, for any frame homomorphism $h : \Omega(Y) \longrightarrow \Omega(X)$, there exists a unique continuous function $f : X \longrightarrow Y$ such that $h = \Omega(f)$.* (See [8], p. 186).

Proof. Let $h : \Omega(Y) \longrightarrow \Omega(X)$ be a frame homomorphism. Let $x \in X$ and consider the cp-filter $\mathcal{U}(x)$ in $\Omega(X)$. Then $h^{-1}[\mathcal{U}(x)]$ is a cp-filter in $\Omega(Y)$. Since Y is sober, there exists $y \in Y$ such that $\mathcal{U}(y) = h^{-1}[\mathcal{U}(x)]$. Let $f : X \longrightarrow Y$ be a map and choose it such that $y = f(x)$. Then $h^{-1}[\mathcal{U}(x)] = \mathcal{U}(f(x))$. Hence, for all $U \in \Omega(Y)$ and all $x \in X$,

$$\begin{aligned}
&(U \in \mathcal{U}(f(x)) \text{ iff } U \in h^{-1}[\mathcal{U}(x)]) \\
\iff &(f(x) \in U \text{ iff } h(U) \in \mathcal{U}(x)) \\
\iff &(x \in f^{-1}[U] \text{ iff } x \in h(U)) \\
\iff &f^{-1}[U] = h(U).
\end{aligned}$$

Therefore f is continuous and $h = \Omega(f)$. Since Y is sober, Y is T_0. Hence by Lemma 4.2.1, for any continuous $g : X \longrightarrow Y$ distinct to f, we have $h = \Omega(f) \neq \Omega(g)$. Therefore such continuous f is indeed unique. $\square$

We have considered the category Top, now let us consider the subcategory Sob of Top; whose objects are sober spaces and whose morphisms are continuous functions. Sob is in fact a full subcategory of Top since, for all $X, Y \in \mathrm{ob}(\mathsf{Sob})$, $\hom_{\mathsf{Sob}}(X, Y) = \hom_{\mathsf{Top}}(X, Y)$. That is to say, the morphisms from X to Y in Sob are precisely the continuous functions from X to Y in Top. Let us restrict the domain of the functor $\Omega : \mathsf{Top} \longrightarrow \mathsf{Frm}$ to the full subcategory Sob so that we have a contravariant functor $\Omega : \mathsf{Sob} \longrightarrow \mathsf{Frm}$. Now let $X, Y \in \mathrm{ob}(\mathsf{Sob})$. In light of Theorem 4.2.3, for all $h \in \hom_{\mathsf{Frm}}(\Omega(Y), \Omega(X))$, there exists a *unique* $f \in \hom_{\mathsf{Sob}}(X, Y)$ such that $h = \Omega(f)$. This means that, under the functor Ω, the function $\hom_{\mathsf{Sob}}(X, Y) \longrightarrow \hom_{\mathsf{Frm}}(\Omega(Y), \Omega(X))$ is both surjective and injective. That is to say, $\Omega : \mathsf{Sob} \longrightarrow \mathsf{Frm}$ is both full and faithful. Therefore $\Omega : \mathsf{Sob} \longrightarrow \mathsf{Frm}$ is a contravariant embedding. Equivalently, $\Omega : \mathsf{Sob} \longrightarrow \mathsf{Loc}$ is a covariant embedding, which enables us to adopt the viewpoint that frames are the generalised (point-free) spaces ([7], p. 11, and [8], p. 190). Thus we may consider point-free topology to be an extension of a significant part classical topology (namely the sober spaces) ([8], p. 187).

4.3 Reconstructing Sober Spaces from Frames

Let Y be a sober space and let P be a one-point space, say $P = \{*\}$ with $\Omega(P) = \{\emptyset, P\} \cong \mathbf{2}$ (where $\mathbf{2} \coloneqq \{0, 1\}$ is the two element chain). Then the points $y \in Y$ are in a one-to-one correspondence with the (continuous) maps $f_y : P \longrightarrow Y$ defined by $f_y(*) = y$ (it is straightforward to verify that such a map is indeed continuous). By Theorem 4.2.3, the continuous maps f_y are in a one-to-one correspondence with the frame homomorphisms $h_y = \Omega(f_y) : \Omega(Y) \longrightarrow \Omega(P)$. Thus the points $y \in Y$ are in a one-to-one correspondence with the frame homomorphisms $h_y = \Omega(f_y)$. Let us define $\mathcal{H} \coloneqq \{h : \Omega(Y) \longrightarrow \Omega(P) \mid h \text{ is a frame homomorphism}\}$. Then we can say that there is a bijection $\varphi : Y \longrightarrow \mathcal{H}$ where, for each $y \in Y$, $\varphi(y) \coloneqq \Omega(f_y)$. For each $U \in \Omega(Y)$ put $\widetilde{U} \coloneqq \{h \in \mathcal{H} : h(U) = P\}$. (See [8], p. 186).

Proposition 4.3.1. $\tau_{\mathcal{H}} \coloneqq \{\widetilde{U} : U \in \Omega(Y)\}$ *is a topology on* $\mathcal{H}$.

Proof. For all $h \in \mathcal{H}$, $h(Y) = P$, hence $\widetilde{Y} = \mathcal{H}$ – which implies that $\mathcal{H} \in \tau_{\mathcal{H}}$. Also, for all $h \in \mathcal{H}$, $h(\emptyset) = \emptyset$ – which means that $\widetilde{\emptyset} = \{h \in \mathcal{H} : h(\emptyset) = P\} = \emptyset$ and hence $\emptyset \in \tau_{\mathcal{H}}$. Let $U, V \in \Omega(Y)$, so that $\tilde{U}, \tilde{V} \in \tau_{\mathcal{H}}$. Then $\tilde{U} \cap \tilde{V} = \{h \in \mathcal{H} : h(U) = P \text{ and } h(V) = P\}$. It follows that, given $h \in \mathcal{H}$, $h \in \tilde{U} \cap \tilde{V}$ iff $h(U) = P$ and $h(V) = P$. Now clearly $h(U) = P$ and $h(V) = P$ implies that $h(U) \cap h(V) = P$. Conversely, suppose that $h(U) \cap h(V) = P$. Then $P \subseteq h(U)$ and $P \subseteq h(V)$. But $P \supseteq A$ for all $A \in \Omega(P)$. Thus $P = h(U)$ and $P = h(V)$. So for all $h \in \mathcal{H}$,

$$\begin{aligned} h \in \tilde{U} \cap \tilde{V} &\iff h(U) = P \text{ and } h(V) = P \\ &\iff h(U) \cap h(V) = P \\ &\iff h(U \cap V) = P \\ &\iff h \in \widetilde{U \cap V}. \end{aligned}$$

Thus $\tilde{U} \cap \tilde{V} = \widetilde{U \cap V}$, which implies that $\tilde{U} \cap \tilde{V} \in \tau_{\mathcal{H}}$. Finally, let $\{U_i\}_{i \in I} \subseteq \Omega(Y)$ so that $\{\tilde{U}_i\}_{i \in I} \subseteq \tau_{\mathcal{H}}$. Then $\bigcup_{i \in I} \tilde{U}_i = \{h \in \mathcal{H} : h(U_i) = P \text{ for some } i \in I\}$. It follows that, given $h \in \mathcal{H}$, $h \in \bigcup_{i \in I} \tilde{U}_i$ iff $h(U_i) = P$ for some $i \in I$. Clearly, if $h(U_i) = P$ for some $i \in I$, then $\bigcup_{i \in I} h(U_i) = P$. Conversely, suppose that $\bigcup_{i \in I} h(U_i) = P$. Now, for each $i \in I$, $h(U_i) = P$ or $h(U_i) = \emptyset$. So it must be that $h(U_i) = P$ for some $i \in I$. So for all $h \in \mathcal{H}$,

$$\begin{aligned} h \in \bigcup_{i \in I} \tilde{U}_i &\iff h(U_i) = P \text{ for some } i \in I \\ &\iff \bigcup_{i \in I} h(U_i) = P \\ &\iff h\Big(\bigcup_{i \in I} U_i\Big) = P \\ &\iff h \in \widetilde{\bigcup_{i \in I} U_i}. \end{aligned}$$

Thus $\bigcup_{i \in I} \tilde{U}_i = \widetilde{\bigcup_{i \in I} U_i}$, which implies that $\bigcup_{i \in I} \tilde{U}_i \in \tau_{\mathcal{H}}$. We can conclude that $(\mathcal{H}, \tau_{\mathcal{H}})$ is a topological space. $\square$

Proposition 4.3.2. $\varphi : Y \longrightarrow \mathcal{H}$ *is a homeomorphism.*

Proof. We already know that φ is a bijection, so we need to show that φ is continuous and an open map. Let $U \in \Omega(Y)$, so that $\tilde{U} \in \tau_{\mathcal{H}}$. Then:

$$\begin{aligned} \varphi^{-1}[\tilde{U}] &= \{y \in Y : \varphi(y) \in \tilde{U}\} \\ &= \{y \in Y : \Omega(f_y) \in \tilde{U}\} \\ &= \{y \in Y : \Omega(f_y)(U) = P\} \\ &= \{y \in Y : f_y^{-1}[U] = P\}. \end{aligned}$$

Note that $f_y^{-1}[U] = P \iff \{p \in P : f_y(p) \in U\} = \{*\} \iff f_y(*) \in U \iff y \in U$. Hence $\varphi^{-1}[\tilde{U}] = \{y \in Y : y \in U\} = U$. Thus φ is continuous. Next consider $\varphi[U] = \{\varphi(y) \in \mathcal{H} : y \in U\} = \{\Omega(f_y) \in \mathcal{H} : y \in U\}$. Now if $y \in U$ then $f_y^{-1}[U] = \{*\} = P$, and if $f_y^{-1}[U] = P$ then $y \in U$. So it follows that $y \in U$ iff $\Omega(f_y)(U) = P$. Hence $\varphi[U] = \{\Omega(f_y) \in \mathcal{H} : \Omega(f_y)(U) = P\} = \widetilde{U}$. Thus φ is an open map. We may conclude that φ is a homeomorphism. $\square$

It is here where we obtain the motivation for pursuing point-free topology. Given any sober space Y we can homeomorphically reconstruct Y from the frame $\Omega(Y)$ as the space $(\mathcal{H}, \tau_{\mathcal{H}})$, that is to say, we do not lose any topological information about sober spaces when we study them from the point-free perspective ([7], p. 9). This gives us an ample supply of interesting topological spaces whose open set lattices contain all the information there is to know about the space (recall that every T_2 space is sober and thus every T_2 space can be homeomorphically reconstructed from its open set lattice).

5 Representation of Points in Frames

We present three ways to represent points in frames (as is given by [7], pp. 13–14). Points in frames will become useful when we wish to recover topological spaces from frames; the set of points in a frame allows us to naturally define a topology on it. For a frame L we will use the following notations throughout the rest of the report:

- $\mathcal{H}(L) \coloneqq \{h : L \longrightarrow \mathbf{2} \mid h \text{ is a frame homomorphism}\}$;
- $\Sigma(L) \coloneqq \{F \subsetneq L : F \text{ is a cp-filter}\}$ and
- $\mathcal{M}(L) \coloneqq \{p \in L : p \text{ is meet-irreducible}\}$.

5.1 Representing Points as Frame Homomorphisms

Recall from 4.3 that, given a sober space Y, the points $y \in Y$ are in a one-to-one correspondence with the frame homomorphisms $h_y : \Omega(Y) \longrightarrow \Omega(P)$ where P is a one-point space. Now $\Omega(P) \cong \mathbf{2}$, so we can say that the points $y \in Y$ are in a one-to-one correspondence with the frame homomorphisms $h_y : \Omega(Y) \longrightarrow \mathbf{2}$. This provides us with a natural definition for a *point* in the frame $\Omega(Y)$.

Definition 5.1.1. Let Y be a sober space. A **point** in the frame $\Omega(Y)$ is a frame homomorphism $h : \Omega(Y) \longrightarrow \mathbf{2}$.

However, we would still like to have a definition of a point in a general frame (not just a frame which is the open set lattice of some sober space). We now modify the preceding definition slightly.

Definition 5.1.2. Let L be a frame. A **point** in L is a frame homomorphism $h : L \longrightarrow \mathbf{2}$.

5.2 Representing Points as Completely Prime Filters

Proposition 5.2.1. *Let L be a frame. Then, for each $h \in \mathcal{H}(L)$, $F_h \coloneqq \{x \in L : h(x) = 1\}$ is a cp-filter in L.*

Proof. Let $h \in \mathcal{H}(L)$. F_h is non-empty because $1 \in F_h$, and $F_h \neq L$ because $0 \notin F_h$. Let $x \in F_h$ and $y \in L$ be such that $x \leqslant y$. Then $x = x \wedge y$ and $1 = h(x) = h(x \wedge y) = h(x) \wedge h(y)$. Thus $1 = 1 \wedge h(y)$, which implies $h(y) = 1$ and hence $y \in F_h$. Let $a, b \in F_h$. Then $h(a \wedge b) = h(a) \wedge h(b) = 1 \wedge 1 = 1$. Thus $a \wedge b \in F_h$. It follows that F_h is a proper filter. We must now show that F_h is completely prime. Let $\{x_i\}_{i \in I} \subseteq L$ be such that $\bigvee_{i \in I} x_i \in F_h$. Then $1 = h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} h(x_i)$. For all $i \in I$, either $h(x_i) = 0$ or $h(x_i) = 1$. Since $\bigvee_{i \in I} h(x_i) = 1$, it must be that there exists $j \in I$ such that $h(x_j) = 1$ and hence $x_j \in F_h$. We may conclude that $F_h \subsetneq L$ is a cp-filter. □

Lemma 5.2.2. *Let L be a frame and let $F \subsetneq L$ be a cp-filter. Then the map $h_F : L \longrightarrow \mathbf{2}$ defined by*

$$h_F(x) \coloneqq \begin{cases} 1 & \text{for } x \in F, \\ 0 & \text{for } x \notin F \end{cases} \tag{5.1}$$

is a frame homomorphism.

Proof. We first show that h preserves arbitrary joins. Let $\{x_i\}_{i \in I} \subseteq L$. Suppose that $\bigvee_{i \in I} x_i \in F$. Then there exists $j \in I$ such that $x_j \in F$, which implies that $1 = h(\bigvee_{i \in I} x_i) = h(x_j)$. But since $h(x_j) = 1$, it follows that $\bigvee_{i \in I} h(x_i) = 1$. Hence $h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} h(x_i)$. Now suppose that $\bigvee_{i \in I} x_i \notin F$. For each $i \in I$ we have $x_i \leqslant \bigvee_{i \in I} x_i$, so if there exists $i \in I$ with $x_i \in F$ then $\bigvee_{i \in I} x_i \in F$. But we have $\bigvee_{i \in I} x_i \notin F$, so it must be that $x_i \notin F$ for each $i \in I$. Then $0 = h(\bigvee_{i \in I} x_i) = h(x_i)$ for each $i \in I$. Thus $\bigvee_{i \in I} h(x_i) = 0$ and hence $h(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} h(x_i)$. It now follows that h preserves arbitrary joins.

Next we show that h preserves binary meets. Let $a, b \in L$. Suppose that $a \wedge b \in F$. Then $h(a \wedge b) = 1$. Since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, it follows that $a, b \in F$. This implies that $1 = h(a) = h(b)$ and hence $h(a \wedge b) = h(a) \wedge h(b)$. Now suppose that $a \wedge b \notin F$. Then (since F is a filter) $a \notin F$ or $b \notin F$. So $h(a) = 0$ or $h(b) = 0$. Then $h(a) \wedge h(b) = 0 = h(a \wedge b)$. It thus follows that h preserves binary meets. □

Proposition 5.2.3. *Let L be a frame and let $\gamma : \mathcal{H}(L) \longrightarrow \Sigma(L)$ be a map defined by $h \longmapsto F_h \coloneqq \{x \in L : h(x) = 1\}$. Then γ is a bijection.*

Proof. Let $F \subsetneq L$ be a cp-filter and consider the frame homomorphism $h_F : L \longrightarrow \mathbf{2}$ as defined by Eq.(5.1). Then $\gamma(h_F) = F_{h_F} = \{x \in L : h_F(x) = 1\} = \{x \in L : x \in F\} = F$. Hence γ is surjective. Next, let $h_1, h_2 : L \longrightarrow \mathbf{2}$ be distinct frame homomorphisms. Then there exists $x \in L$ such that $h_1(x) \neq h_2(x)$. Without loss of generality, suppose that $h_1(x) = 1$ and $h_2(x) = 0$. Then $x \in \gamma(h_1)$ and $x \notin \gamma(h_2)$, which implies that $\gamma(h_1) \neq \gamma(h_2)$. Hence γ is injective. □

From what we have shown, it follows that, for a frame L, the frame homomorphisms $h : L \longrightarrow \mathbf{2}$ are in a one-to-one correspondence with the cp-filters $F \subsetneq L$. In this way we can represent points in frames as cp-filters. Indeed it is natural to regard points in frames as cp-filters; points in a space can be represented by their family of open neighbourhoods (a cp-filter) and for sober spaces these are the only cp-filters.

5.3 Representing Points as Meet-Irreducible Elements

Proposition 5.3.1. *Let L be a frame and $h \in \mathcal{H}(L)$. Then $p_h \coloneqq \bigvee\{x \in L : h(x) = 0\}$ is meet-irreducible.*

Proof. Put $A \coloneqq \{x \in L : h(x) = 0\}$, then $p_h = \bigvee A$. Note that:

$$\begin{aligned}(\forall x \in A)\ \ h(x) = 0 &\iff \bigvee\{h(x) : x \in A\} = 0 \\ &\iff h(\bigvee\{x : x \in A\}) = 0 \\ &\iff h(\bigvee A) = 0 \\ &\iff \bigvee A \in A \\ &\iff p_h \in A.\end{aligned}$$

Since $h(1) = 1$, it follows that $1 \notin A$ and hence $p_h \neq 1$. Let $a, b \in L$ and suppose that $a \wedge b \leqslant p_h$. Then $h(a \wedge b) \leqslant h(p_h)$ (because h is order-preserving) which implies that $h(a) \wedge h(b) \leqslant 0$ and hence $h(a) \wedge h(b) = 0$. Thus $h(a) = 0$ or $h(b) = 0$. Suppose that it is $h(a) = 0$, then $a \in A$ and $a \leqslant \bigvee A = p_h$. We may conclude that p_h is meet-irreducible. □

Lemma 5.3.2. *Let L be a frame and $p \in \mathcal{M}(L)$. Let $h_p : L \longrightarrow \mathbf{2}$ be a map defined by*

$$h_p(x) \coloneqq \begin{cases} 0 & \text{for } x \leqslant p, \\ 1 & \text{for } x \nleqslant p. \end{cases} \tag{5.2}$$

Then h_p is a frame homomorphism.

Proof. We first show that h_p preserves arbitrary joins. Let $\{x_i\}_{i \in I} \subseteq L$. Suppose that $\bigvee_{i \in I} x_i \leqslant p$. Then $x_i \leqslant p$ for each $i \in I$, which implies that $h_p(x_i) = 0$ for each $i \in I$. Thus $\bigvee_{i \in I} h_p(x_i) = 0 = h_p(\bigvee_{i \in I} x_i)$. Now suppose that $\bigvee_{i \in I} x_i \nleqslant p$. If $x_i \leqslant p$ for each $i \in I$, then p is an upper bound of $\{x_i\}_{i \in I}$ and hence $\bigvee_{i \in I} x_i \leqslant p$ – a contradiction. It must be that there exists $j \in I$ such that $x_j \nleqslant p$ and thus $h_p(x_j) = 1$. Therefore $\bigvee_{i \in I} h_p(x_i) = 1 = h_p(\bigvee_{i \in I} x_i)$. It follows that h_p preserves arbitrary joins.

Next we show that h_p preserves binary meets. Let $a, b \in L$. Suppose that $a \wedge b \leqslant p$. Then $a \leqslant p$ or $b \leqslant p$ (since p is meet-irreducible). Assume that it is $a \leqslant p$. Then $h_p(a) = 0 \leqslant h_p(b)$, so $h_p(a) \wedge h_p(b) = 0 = h_p(a \wedge b)$. Now suppose that $a \wedge b \nleqslant p$. If $a \leqslant p$ or $b \leqslant p$ then, since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, $a \wedge b \leqslant p$ – a contradiction. So it must be that $a \nleqslant p$ and $b \nleqslant p$, hence $h_p(a) = h_p(b) = 1$. Thus $h_p(a) \wedge h_p(b) = 1 = h_p(a \wedge b)$. It follows that h_p preserves binary meets. □

Proposition 5.3.3. *Let L be a frame and let $\delta : \mathcal{H}(L) \longrightarrow \mathcal{M}(L)$ be a map defined by $h \longmapsto p_h \coloneqq \bigvee\{x \in L : h(x) = 0\}$. Then δ is bijective.*

Proof. Let $h, g \in \mathcal{H}(L)$ be distinct. Then there exists $q \in L$ such that $h(q) \neq g(q)$. Without loss of generality, suppose that $h(q) = 1$ and $g(q) = 0$. Put $A \coloneqq \{x \in L : h(x) = 0\}$ and $B \coloneqq \{x \in L : g(x) = 0\}$. Then $q \notin A$ and $q \in B$, so $A \neq B$. As was similarly shown in the proof of Proposition 5.3.1, $\bigvee A \in A$ and $\bigvee B \in B$. Now $h(\bigvee B) = \bigvee\{h(x) : x \in B\} = 1$ (since $q \in B$ and $h(q) = 1$). This implies that $\bigvee B \notin A$ and hence $\bigvee B \neq \bigvee A$. Thus $\delta(g) \neq \delta(h)$, and it follows that δ is injective. Let $p \in \mathcal{M}(L)$ and let $h_p : L \longrightarrow \mathbf{2}$ be the frame homomorphism as defined in Eq.(5.2). Clearly p is an upper bound of $\{x \in L : h_p(x) = 0\}$ and $p \in \{x \in L : h_p(x) = 0\}$, hence p is the top of $\{x \in L : h_p(x) = 0\}$. That is to say, $p = \bigvee\{x \in L : h_p(x) = 0\} \eqqcolon \delta(h_p)$. It follows that δ is surjective. □

We may conclude that, for a frame L, the frame homomorphisms $h : L \longrightarrow \mathbf{2}$ are in a one-to-one correspondence with the meet-irreducible elements $p \in \mathcal{M}(L)$. In this way, we may represent points in frames as meet-irreducible elements.

6 The Spectrum Adjunction

By this stage we are comfortable with the process of starting with a topological space X and then obtaining a frame from X as the open set lattice $\Omega(X)$. We now consider the reverse of this process. Starting with a frame L, how may we obtain a topological space from L? This is the question we seek to answer.

6.1 The Spectrum: Recovering a Topological Space from a Frame

We know that we can represent points in a frame L as the cp-filters in L, we then simply need to define a topology on $\Sigma(L)$.

Proposition 6.1.1. *Let L be a frame and for each $a \in L$ define $\Sigma_a \coloneqq \{F \in \Sigma(L) : a \in F\}$. Put $\mathcal{T}_{\Sigma(L)} \coloneqq \{\Sigma_a : a \in L\}$. Then $\mathcal{T}_{\Sigma(L)}$ is a topology on $\Sigma(L)$.*

Proof. The only filter in L which contains 0 is just the entire frame L (any filter is an up-set, so if it contains 0 then it contains every element in L). Since cp-filters do not coincide with L, it follows that there are no cp-filters which contain 0: $\Sigma_0 = \{F \in \Sigma(L) : 0 \in F\} = \emptyset$. Thus $\emptyset \in \mathcal{T}_{\Sigma(L)}$. Every (cp-)filter in L is an up-set, so every cp-filter in L contains 1: $\Sigma_1 = \{F \in \Sigma(L) : 1 \in F\} = \Sigma(L)$. Thus $\Sigma(L) \in \mathcal{T}_{\Sigma(L)}$. Let $a, b \in L$ and consider

$$\begin{aligned}
\Sigma_a \cap \Sigma_b &= \{F \in \Sigma(L) : a, b \in F\} \\
&= \{F \in \Sigma(L) : a \wedge b \in F\} \ \ \text{(since, for a (cp-)filter } F,\ a, b \in F \text{ iff } a \wedge b \in F) \\
&= \Sigma_{a \wedge b}.
\end{aligned}$$

Thus $\Sigma_a \cap \Sigma_b \in \mathcal{T}_{\Sigma(L)}$. Let $\{a_i\}_{i \in I} \subseteq L$. Note that for a cp-filter F in L, if there exists $i \in I$ such that $a_i \in F$ then $\bigvee_{i \in I} a_i \in F$ (since F is an up-set), conversely, if $\bigvee_{i \in I} a_i \in F$ then there exists $i \in I$ such that $a_i \in F$ (since F is completely prime). So we have

$$\begin{aligned}
\bigcup_{i \in I} \Sigma_{a_i} &= \{F \in \Sigma(L) : (\exists i \in I)\ F \in \Sigma_{a_i}\} \\
&= \{F \in \Sigma(L) : (\exists i \in I)\ a_i \in F\} \\
&= \{F \in \Sigma(L) : \bigvee_{i \in I} a_i \in F\} \\
&= \Sigma_{\bigvee_{i \in I} a_i}.
\end{aligned}$$

Thus $\bigcup_{i \in I} \Sigma_{a_i} \in \mathcal{T}_{\Sigma(L)}$. We may conclude that $\mathcal{T}_{\Sigma(L)}$ is a topology on $\Sigma(L)$. ☐

It follows that we may regard $\Sigma(L)$ as a space with topology $\mathcal{T}_{\Sigma(L)}$ – we call the space $\Sigma(L)$ the *spectrum* of L ([7], p. 15). Recall from Lemma 4.2.2 that if $L \xrightarrow{h} M$ is a frame homomorphism and $F \subsetneq M$ is a cp-filter, then $h^{-1}[F]$ is a cp-filter.

Definition 6.1.2. Let $L \xrightarrow{h} M$ be a frame homomorphism. We define the map $\Sigma(L) \xleftarrow{\Sigma(h)} \Sigma(M)$ by $\Sigma(h)(F) \coloneqq h^{-1}[F]$.

Proposition 6.1.3. *Let $L \xrightarrow{h} M$ be a frame homomorphism. Then the map $\Sigma(L) \xleftarrow{\Sigma(h)} \Sigma(M)$ is continuous.* (See [8], p. 188).

Proof. Let $a \in L$ and consider $\Sigma_a \in \mathfrak{T}_{\Sigma(L)}$. We have

$$\begin{aligned}(\Sigma(h))^{-1}[\Sigma_a] &= \{F \in \Sigma(M) : \Sigma(h)(F) \in \Sigma_a\} \\ &= \{F \in \Sigma(M) : h^{-1}[F] \in \Sigma_a\} \\ &= \{F \in \Sigma(M) : a \in h^{-1}[F]\} \\ &= \{F \in \Sigma(M) : h(a) \in F\} \\ &= \Sigma_{h(a)} \in \mathfrak{T}_{\Sigma(M)}.\end{aligned}$$

We may conclude that $\Sigma(h)$ is continuous. □

6.1.1 An Observation and the Functor Σ

We note that we can view Σ as a mapping $\mathrm{ob}(\mathsf{Frm}) \longrightarrow \mathrm{ob}(\mathsf{Top})$, where any frame L is mapped to its spectrum $\Sigma(L)$, $L \longmapsto \Sigma(L)$. We can also view Σ as sending each frame homomorphism $L \xrightarrow{h} M$ to a continuous function $\Sigma(L) \xleftarrow{\Sigma(h)} \Sigma(M)$, that is, for each $L, M \in \mathrm{ob}(\mathsf{Frm})$, we can view Σ as a mapping $\mathrm{hom}_{\mathsf{Frm}}(L, M) \longrightarrow \mathrm{hom}_{\mathsf{Top}}(\Sigma(M), \Sigma(L))$.

Proposition 6.1.4. *Let $L \xrightarrow{h} M \xrightarrow{g} N$ be frame homomorphisms. Then $\Sigma(gh) = \Sigma(h)\Sigma(g)$.*

Proof. Let $F \in \Sigma(N)$. Then

$$\begin{aligned}\Sigma(gh)(F) &= (gh)^{-1}[F] \\ &= h^{-1}[g^{-1}[F]] \\ &= \Sigma(h)(\Sigma(g)(F)) \\ &= (\Sigma(h)\Sigma(g))(F).\end{aligned}$$

Thus $\Sigma(gh) = \Sigma(h)\Sigma(g)$. □

Proposition 6.1.5. *Let L be a frame and let 1_L be the identity map on L. Then $\Sigma(1_L) = 1_{\Sigma(L)}$.*

Proof. Let $F \in \Sigma(L)$. Then $\Sigma(1_L)(F) = 1_L^{-1}[F] = F = 1_{\Sigma(L)}(F)$. □

We may conclude that we have a *contravariant* functor $\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Top}$. We've introduced the category Loc as the dual of Frm thus, bearing in mind the observation 2.5.1, we may view Σ as a *covariant* functor $\Sigma : \mathsf{Loc} \longrightarrow \mathsf{Top}$. As a covariant functor, Σ sends each morphism $M \xrightarrow{h} L$ in Loc to a continuous function $\Sigma(M) \xrightarrow{\Sigma(h)} \Sigma(L)$ in Top.

6.2 An Adjunction between Top and Loc

We now have a means of moving between the categories Top and Loc, we have a pair of covariant functors $\mathsf{Top} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathsf{Loc}$. We will show that Σ is in fact a right adjoint to Ω.

6.2.1 The Natural Transformation $\sigma : \Omega\Sigma \Longrightarrow 1_{\mathsf{Loc}}$

For any frame L let $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ be a map defined by $\varepsilon_L(a) \coloneqq \Sigma_a$. Then, recalling the proof of Proposition 6.1.1, ε_L is a frame homomorphism:

$$\varepsilon_L(a \wedge b) = \Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b = \varepsilon_L(a) \wedge \varepsilon_L(b), \text{ and}$$
$$\varepsilon_L(\bigvee_{i\in I} a_i) = \Sigma_{\bigvee_{i\in I} a_i} = \bigcup_{i\in I} \Sigma_{a_i} = \bigvee_{i\in I} \varepsilon_L(a_i).$$

Note that we may equivalently view the frame homomorphism $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ as a morphism $\sigma_L : \Omega\Sigma(L) \longrightarrow L$ in Loc. For the moment let us view the functors Ω and Σ contravariantly, that is, as functors $\Omega : \mathsf{Top} \longrightarrow \mathsf{Frm}$ and $\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Top}$. Then their composition $\Omega\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Frm}$ is a *covariant* functor.

Proposition 6.2.1. *The collection*

$$\varepsilon \coloneqq \{L \xrightarrow{\varepsilon_L} \Omega\Sigma(L) : L \in \mathrm{ob}(\mathsf{Frm})\}$$

constitutes a natural transformation $\varepsilon : 1_{\mathsf{Frm}} \Longrightarrow \Omega\Sigma$. (cf. [8], p. 188).

Proof. Let $L \xrightarrow{h} M$ be a frame homomorphism. We need to verify that the diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\varepsilon_L"'] & M \arrow[d, "\varepsilon_M"] \\ \Omega\Sigma(L) \arrow[r, "\Omega\Sigma(h)"'] & \Omega\Sigma(M) \end{tikzcd}$$

commutes. Let $a \in L$ and consider

$$\begin{aligned} \Omega\Sigma(h)(\varepsilon_L(a)) &= \Omega\Sigma(h)(\Sigma_a) \\ &= (\Sigma(h))^{-1}[\Sigma_a] \\ &= \Sigma_{h(a)} \ \text{ (recall the proof of Proposition 6.1.3)} \\ &= \varepsilon_M(h(a)). \end{aligned}$$

Hence $(\Omega\Sigma(h))\varepsilon_L = \varepsilon_M h$ and we may conclude that the diagram commutes. Therefore $\varepsilon : 1_{\mathsf{Frm}} \Longrightarrow \Omega\Sigma$ is a natural transformation. $\square$

Let us consider the collection

$$\sigma \coloneqq \{\Omega\Sigma(L) \xrightarrow{\sigma_L} L : L \in \mathrm{ob}(\mathsf{Loc})\} \tag{6.1}$$

of morphisms in Loc. As a result of Proposition 6.2.1 and the duality principle, for each morphism $M \xrightarrow{h} L$ in Loc, the diagram (in Loc)

$$\begin{tikzcd} L & M \arrow[l, "h"'] \\ \Omega\Sigma(L) \arrow[u, "\sigma_L"] & \Omega\Sigma(M) \arrow[l, "\Omega\Sigma(h)"] \arrow[u, "\sigma_M"'] \end{tikzcd} \tag{6.2}$$

commutes. This diagram (6.2) may equivalently be drawn as

$$\begin{tikzcd} \Omega\Sigma(M) \arrow[r, "\Omega\Sigma(h)"] \arrow[d, "\sigma_M"'] & \Omega\Sigma(L) \arrow[d, "\sigma_L"] \\ M \arrow[r, "h"'] & L \end{tikzcd} \tag{6.3}$$

Thus for each morphism $M \xrightarrow{h} L$ in Loc, the diagram (6.3) commutes, and hence the collection (6.1) constitutes a natural transformation $\sigma : \Omega\Sigma \Longrightarrow 1_{\mathsf{Loc}}$.

6.2.2 The Natural Transformation $\rho : 1_{\mathsf{Top}} \implies \Sigma\Omega$

For any space X we have $\Sigma\Omega(X) \coloneqq \{\mathcal{F} \subsetneq \Omega(X) : \mathcal{F} \text{ is a cp-filter}\}$, and define a map $\rho_X : X \longrightarrow \Sigma\Omega(X)$ by $\rho_X(x) \coloneqq \mathcal{U}(x)$ (recall from Proposition 4.1.10 that $\mathcal{U}(x)$ is a cp-filter in $\Omega(X)$ for each $x \in X$).

Remark 6.2.2. *Let X be a space. Then ρ_X is continuous.* (See [8], p. 188).

Proof. Let $U \in \Omega(X)$ and consider $\Sigma_U \in \mathcal{T}_{\Sigma\Omega(X)}$. Then

$$\begin{aligned}
\rho_X^{-1}[\Sigma_U] &= \{x \in X : \mathcal{U}(x) \in \Sigma_U\} \\
&= \{x \in X : U \in \mathcal{U}(x)\} \\
&= \{x \in X : x \in U\} \\
&= U.
\end{aligned}$$

□

Note that we have contravariant functors $\mathsf{Top} \xrightarrow{\Omega} \mathsf{Frm} \xrightarrow{\Sigma} \mathsf{Top}$, and thus the composition $\mathsf{Top} \xrightarrow{\Sigma\Omega} \mathsf{Top}$ is a covariant functor.

Proposition 6.2.3. *The collection*

$$\rho \coloneqq \{X \xrightarrow{\rho_X} \Sigma\Omega(X) : X \in \mathrm{ob}(\mathsf{Top})\}$$

constitutes a natural transformation $\rho : 1_{\mathsf{Top}} \implies \Sigma\Omega$. (See [8], p. 188).

Proof. Let $X \xrightarrow{f} Y$ be a continuous function. We need to verify that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_X"'] & Y \arrow[d, "\rho_Y"] \\
\Sigma\Omega(X) \arrow[r, "\Sigma\Omega(f)"'] & \Sigma\Omega(Y)
\end{tikzcd}$$

commutes. Let $x \in X$ and consider

$$\begin{aligned}
\Sigma\Omega(f)(\rho_X(x)) &= \Sigma\Omega(f)(\mathcal{U}(x)) \\
&= (\Omega(f))^{-1}[\mathcal{U}(x)] \\
&= \{U \in \Omega(Y) : \Omega(f)(U) \in \mathcal{U}(x)\} \\
&= \{U \in \Omega(Y) : f^{-1}[U] \in \mathcal{U}(x)\} \\
&= \{U \in \Omega(Y) : x \in f^{-1}[U]\} \\
&= \{U \in \Omega(Y) : f(x) \in U\} \\
&= \mathcal{U}(f(x)) \\
&= \rho_Y(f(x)).
\end{aligned}$$

Thus $(\Sigma\Omega(f))\rho_X = \rho_Y f$ and we may conclude that the diagram commutes. Therefore $\rho : 1_{\mathsf{Top}} \implies \Sigma\Omega$ is a natural transformation. □

6.2.3 The Spectrum Adjunction

Theorem 6.2.4. *The functor* $\Sigma : \mathsf{Loc} \longrightarrow \mathsf{Top}$ *is right adjoint to the functor* $\Omega : \mathsf{Top} \longrightarrow \mathsf{Loc}$, *with* $\rho : 1_{\mathsf{Top}} \implies \Sigma\Omega$ *the unit and* $\sigma : \Omega\Sigma \implies 1_{\mathsf{Loc}}$ *the counit of the adjunction.* (See [9], pp. 59–60).

Proof. Firstly we need to verify that the triangle

$$\begin{tikzcd}
\Sigma(L) \arrow[r, "\rho_{\Sigma(L)}"] \arrow[dr, "1_{\Sigma(L)}"'] & \Sigma\Omega\Sigma(L) \arrow[d, "\Sigma(\sigma_L)"] \\
& \Sigma(L)
\end{tikzcd}$$

commutes for all frames L. Now, for each frame L, $\Sigma(\sigma_L) = \Sigma(\varepsilon_L)$. Hence we need to verify that the triangle

$$\begin{tikzcd}
\Sigma(L) \arrow[r, "\rho_{\Sigma(L)}"] \arrow[dr, "1_{\Sigma(L)}"'] & \Sigma\Omega\Sigma(L) \arrow[d, "\Sigma(\varepsilon_L)"] \\
& \Sigma(L)
\end{tikzcd} \tag{6.4}$$

commutes for all frames L. Let L be a frame and let $F \in \Sigma(L)$. Consider

$$\begin{aligned}
\Sigma(\varepsilon_L)(\rho_{\Sigma(L)}(F)) &= \Sigma(\varepsilon_L)(\mathcal{U}(F)) \\
&= \varepsilon_L^{-1}[\mathcal{U}(F)] \\
&= \{x \in L : \varepsilon_L(x) \in \mathcal{U}(F)\} \\
&= \{x \in L : \Sigma_x \in \mathcal{U}(F)\} \\
&= \{x \in L : F \in \Sigma_x\} \\
&= \{x \in L : x \in F\} \\
&= F.
\end{aligned}$$

Therefore $\Sigma(\varepsilon_L)\rho_{\Sigma(L)} = 1_{\Sigma(L)}$, and it follows that the triangle (6.4) commutes. Secondly we need to verify that, for all spaces X, the triangle (viewed as in Loc)

$$\begin{tikzcd}
\Omega(X) \arrow[r, "\Omega(\rho_X)"] \arrow[dr, "1_{\Omega(X)}"'] & \Omega\Sigma\Omega(X) \arrow[d, "\sigma_{\Omega(X)}"] \\
& \Omega(X)
\end{tikzcd}$$

commutes. Equivalently we need to verify that, for all spaces X, the triangle (viewed as in Frm)

$$\begin{tikzcd}
\Omega(X) & \Omega\Sigma\Omega(X) \arrow[l, "\Omega(\rho_X)"'] \\
& \Omega(X) \arrow[ul, "1_{\Omega(X)}"] \arrow[u, "\varepsilon_{\Omega(X)}"']
\end{tikzcd} \tag{6.5}$$

commutes. Let X be a space and let $U \in \Omega(X)$. Consider

$$\begin{aligned}\Omega(\rho_X)(\varepsilon_{\Omega(X)}(U)) &= \Omega(\rho_X)(\Sigma_U) \\ &= \rho_X^{-1}[\Sigma_U] \\ &= \{x \in X : \rho_X(x) \in \Sigma_U\} \\ &= \{x \in X : \mathcal{U}(x) \in \Sigma_U\} \\ &= \{x \in X : U \in \mathcal{U}(x)\} \\ &= \{x \in X : x \in U\} \\ &= U.\end{aligned}$$

Therefore $\Omega(\rho_X)\varepsilon_{\Omega(X)} = 1_{\Omega(X)}$ and it follows that the triangle (6.5) commutes. We may now conclude that Σ is right adjoint to Ω, with $\rho : 1_{\mathsf{Top}} \Longrightarrow \Sigma\Omega$ the unit and $\sigma : \Omega\Sigma \Longrightarrow 1_{\mathsf{Loc}}$ the counit of the adjunction. □

This adjunction between the categories Top and Loc is known as the *spectrum adjunction*. For further information on the spectrum adjunction, see [7] (pp. 15–18); [8] (pp. 188–189) and [9] (pp. 59–60).

7 Spatial Frames

We will call any frame L which resembles an open set lattice $\Omega(X)$ *spatial*. That is to say, spatial frames are those frames which can be realised as the open set lattices of topological spaces. More precisely, we will say that a frame L is spatial if it is isomorphic to $\Omega(X)$ for some space X. Already, any frame L is intimately connected with the open set lattice of a topological space, namely the open set lattice of its spectrum $\Sigma(L)$: recall that we have a homomorphism $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ (we will see that ε_L is in fact surjective). In this section, we present a complete lattice-theoretic characterisation of spatial frames.

7.1 Spatiality

Definition 7.1.1 (Spatiality)**.** A frame L is said to be **spatial** iff there exists a space X such that $L \cong \Omega(X)$ ([7], p. 10).

Remark 7.1.2. *For any frame L, ε_L is surjective.*

Proof. Let $U \in \Omega\Sigma(L)$. Then $U = \Sigma_a$ for some $a \in L$. But $\Sigma_a = \varepsilon_L(a)$. □

Proposition 7.1.3. *Let X be a space. Then $\varepsilon_{\Omega(X)} : \Omega(X) \longrightarrow \Omega\Sigma\Omega(X)$ is an isomorphism.*

Proof. By the previous remark, it is enough for us to show that $\varepsilon_{\Omega(X)}$ is injective. Let $U, V \in \Omega(X)$ and assume $\varepsilon_{\Omega(X)}(U) = \varepsilon_{\Omega(X)}(V)$. Then $\Sigma_U = \Sigma_V$. This means that, given any cp-filter $F \in \Sigma\Omega(X)$, $U \in F$ iff $V \in F$. Now let $x \in X$ be such that $x \in U$. Then $U \in \mathcal{U}(x)$ (recall that $\mathcal{U}(x)$ is a cp-filter). Hence $U \in \mathcal{U}(x)$ iff $V \in \mathcal{U}(x)$, which implies $x \in U$ iff $x \in V$, thus $U = V$. □

Theorem 7.1.4. *A frame L is spatial iff $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ is injective.* (See [8], p. 189).

Proof. ($\Leftarrow$) Let L be a frame such that $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ is injective. Then (by Remark 7.1.2) ε_L is bijective and hence an isomorphism. Thus L is spatial.

($\Rightarrow$) Let L be a spatial frame. Then there exists a space X and an isomorphism $L \xrightarrow{\varphi} \Omega(X)$. Now recall that $\Omega\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Frm}$ is a covariant functor, hence we have an isomorphism $\Omega\Sigma(L) \xrightarrow{\Omega\Sigma(\varphi)} \Omega\Sigma\Omega(X)$. Consider the diagram (in Frm)

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\varepsilon_L"'] & \Omega(X) \arrow[d, "\varepsilon_{\Omega(X)}"] \\ \Omega\Sigma(L) \arrow[r, "\Omega\Sigma(\varphi)"'] & \Omega\Sigma\Omega(X) \end{tikzcd}$$

Now φ and $\varepsilon_{\Omega(X)}$ are both isomorphisms, thus their composition $\varepsilon_{\Omega(X)}\varphi$ is an isomorphism. Note that this diagram commutes: let $a \in L$ and consider

$$\begin{aligned} (\Omega\Sigma(\varphi)\varepsilon_L)(a) &= \Omega\Sigma(\varphi)(\varepsilon_L(a)) \\ &= \Omega\Sigma(\varphi)(\Sigma_a) \\ &= (\Sigma(\varphi))^{-1}[\Sigma_a] \\ &= \Sigma_{\varphi(a)} \ \text{ (recall the proof of Proposition 6.1.3)} \\ &= \varepsilon_{\Omega(X)}(\varphi(a)) \\ &= (\varepsilon_{\Omega(X)}\varphi)(a). \end{aligned}$$

Thus $\Omega\Sigma(\varphi)\varepsilon_L = \varepsilon_{\Omega(X)}\varphi$ and the diagram indeed commutes. Moreover $\varepsilon_L = (\Omega\Sigma(\varphi))^{-1}\varepsilon_{\Omega(X)}\varphi$, hence $L \xrightarrow{\varepsilon_L} \Omega\Sigma(L)$ is an isomorphism. Therefore ε_L is injective. □

Some authors, including Johnstone [3, 4], take the statement of Theorem 7.1.4 as the *definition* for spatiality.

7.2 A Characterisation Theorem for Spatial Frames

We now present a lattice-theoretic characterisation for spatial frames that makes no reference to topological spaces whatsoever. But first, we take note of the following observation.

Observation 7.2.1. Let L be a frame. We previously showed that the frame homomorphisms $h : L \longrightarrow \mathbf{2}$ are in a one-to-one correspondence with the cp-filters $F \subsetneq L$, where $h \longmapsto F_h := \{x \in L : h(x) = 1\}$. We also showed that the meet-irreducible elements p of L are in a one-to-one correspondence with the frame homomorphisms $h : L \longrightarrow \mathbf{2}$, where $p \longmapsto h_p$ and $h_p(x) = 1$ iff $x \nleqslant p$. It therefore follows that the meet-irreducible elements p of L are in a one-to-one correspondence with the cp-filters $F \subsetneq L$, where $p \longmapsto F_p := F_{h_p} = \{x \in L : h_p(x) = 1\} = \{x \in L : x \nleqslant p\}$.

Recall that, for a frame L, we denote the collection of meet-irreducible elements in L by $\mathcal{M}(L)$.

Theorem 7.2.2. *Let L be a frame. Then L is spatial iff* $(\forall a \in L)$ $a = \bigwedge\{p \in \mathcal{M}(L) : a \leqslant p\}$. (See [7], pp. 18–19).

Proof. ($\Rightarrow$) Suppose that L is spatial. Then $\varepsilon_L : L \longrightarrow \Omega\Sigma(L)$ is injective. Let $a \in L$ and put $A := \{p \in \mathcal{M}(L) : a \leqslant p\}$. Clearly a is a lower bound for A, and hence $a \leqslant \bigwedge A$. Let $b \in L$ be such that $a < b$ (i.e. $a \leqslant b$ and $a \neq b$). Then $\Sigma_a \neq \Sigma_b$ by injectivity of ε_L and, since ε_L is a homomorphism (hence order-preserving), $\Sigma_a \subseteq \Sigma_b$, therefore $\Sigma_a \subsetneq \Sigma_b$. Then there exists a cp-filter $F \subsetneq L$ such that $b \in F$ and $a \notin F$. Thus by Observation 7.2.1 there exists $p \in \mathcal{M}(L)$ such that $b \nleqslant p$ and $a \leqslant p$, viz. there exists $p \in A$ such that $b \nleqslant p$. Therefore b is not a lower bound for A. It follows that a is the greatest lower bound of A: $a = \bigwedge A$.

($\Leftarrow$) Suppose $x = \bigwedge\{p \in \mathcal{M}(L) : x \leqslant p\}$ for each $x \in L$. Let $a, b \in L$ and suppose $\Sigma_a = \Sigma_b$. Then, given any cp-filter $F \subsetneq L$, $a \in F$ iff $b \in F$. By Observation 7.2.1 we can say that, given any $p \in \mathcal{M}(L)$, $a \nleqslant p$ iff $b \nleqslant p$, which is equivalent to saying that $a \leqslant p$ iff $b \leqslant p$. Hence $\{p \in \mathcal{M}(L) : a \leqslant p\} = \{p \in \mathcal{M}(L) : b \leqslant p\}$ and thus:

$$a = \bigwedge\{p \in \mathcal{M}(L) : a \leqslant p\} = \bigwedge\{p \in \mathcal{M}(L) : b \leqslant p\} = b.$$

It follows that ε_L is injective and L is spatial. □

8 Ideals

Ideals play a significant role in algebra in general ([2], p. 44). For example, in lattice theory prime ideals are used in the representation theory of distributive lattices ([2], ch. 11). For a frame L, the collection of all ideals in L, when ordered by inclusion, constitutes a frame (known as the ideal lattice of L). A natural problem to then investigate is the spatiality of the ideal lattice of a frame. Specifically we seek an answer to the question: "under what conditions is the ideal lattice of a frame spatial?". Thus we are looking for a sufficient condition for the ideal lattice of a frame to be spatial.

8.1 What is an Ideal?

Up until now, we have been working with filters. We now introduce the dual notion of a filter, this is an *ideal*.

Definition 8.1.1 (Ideal)**.** Let L be a lattice and J a non-empty subset of L. We say that J is an **ideal** iff:

i. $a, b \in J \implies a \vee b \in J$;

ii. for all $a \in L$ and all $b \in J$, $a \leqslant b \implies a \in J$.

Concisely, an ideal J in a lattice L is a non-empty down-set in L that is closed under finite join. Trivially, the entire lattice L is an ideal. Any ideal J in L for which $J \neq L$ is called *proper*. The collection of all ideals in L is denoted by $\mathcal{J}(L)$ – we order $\mathcal{J}(L)$ by set inclusion.

Example 8.1.2. Let L be a lattice and let $a \in L$. Consider the down-set $\downarrow a := \{x \in L : x \leqslant a\}$. Then $\downarrow a$ is an ideal:

- $\downarrow a$ is non-empty since $a \in \downarrow a$;
- if $x, y \in \downarrow a$, then a is an upper bound of the set $\{x, y\}$ and hence $x \vee y \leqslant a$ (i.e. $x \vee y \in \downarrow a$);
- if $x \in \downarrow a$ and $y \in L$ with $y \leqslant x$, then (since $x \leqslant a$) $y \leqslant a$, hence $y \in \downarrow a$.

Any ideal in L of the form $\downarrow a$ is called *principal*.

8.2 Topped $\bigcap$-Structures

One of our goals is to show that, for a frame L, $\mathfrak{J}(L)$ is a frame. Thus we need to show that $\mathfrak{J}(L)$ is a complete lattice. As an aid for helping us show that $\mathfrak{J}(L)$ is a complete lattice, we introduce the notion of a *topped $\bigcap$-structure* (topped $\bigcap$-structures are explored in [2], pp. 48–49).

Definition 8.2.1 (Topped $\bigcap$-structure)**.** Let X be a set and let $\mathcal{L}$ be a non-empty family of subsets of X ordered by inclusion. We say that $\mathcal{L}$ is a **topped $\bigcap$-structure** on X iff:

i. $X \in \mathcal{L}$; and

ii. for any non-empty family $\{A_i\}_{i\in I} \subseteq \mathcal{L}$, $\bigcap_{i\in I} A_i \in \mathcal{L}$.

Proposition 8.2.2. *Let X be a set and let $\mathcal{L}$ be a topped $\bigcap$-structure on X. Then $\mathcal{L}$ is a complete lattice in which, for any non-empty family $\{A_i\}_{i\in I} \subseteq \mathcal{L}$,*

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i \ \text{ and } \ \bigvee_{i\in I} A_i = \bigcap\{B \in \mathcal{L} : \bigcup_{i\in I} A_i \subseteq B\}.$$

Proof. To show that $\mathcal{L}$ is a complete lattice, we must show that $\mathcal{L}$ has a top and that the meet of every non-empty subset of $\mathcal{L}$ exists in $\mathcal{L}$ (see [2] ch. 2). Indeed, $X \in \mathcal{L}$ (the top element) and, for any non-empty family $\{A_i\}_{i\in I} \subseteq \mathcal{L}$, $\bigcap_{i\in I} A_i \in \mathcal{L}$. Hence $\bigwedge_{i\in I} A_i$ exists in $\mathcal{L}$ and equals $\bigcap_{i\in I} A_i$. Thus $\mathcal{L}$ is a complete lattice. Now $\{A_i\}_{i\in I}$ has at least one upper bound in $\mathcal{L}$, namely X, thus we can write

$$\bigvee_{i\in I} A_i = \bigwedge\{A_i : i \in I\}^u = \bigcap\{A_i : i \in I\}^u$$

(where $\{A_i : i \in I\}^u$ is the set of all upper bounds of $\{A_i\}_{i\in I}$). Now

$$\begin{aligned}\{A_i : i \in I\}^u &= \{B \in \mathcal{L} : (\forall i \in I)\ A_i \subseteq B\}\\ &= \{B \in \mathcal{L} : \bigcup_{i\in I} A_i \subseteq B\}.\end{aligned}$$

Hence $\bigvee_{i\in I} A_i = \bigcap\{B \in \mathcal{L} : \bigcup_{i\in I} A_i \subseteq B\}$. $\square$

Proposition 8.2.3. *Let L be a lattice with bottom 0. Then $\mathfrak{J}(L)$ is a topped $\bigcap$-structure on L.*

Proof. L is trivially an ideal, so $L \in \mathfrak{J}(L)$. Let $\{J_i\}_{i\in I} \subseteq L$ be non-empty. We must show that $\bigcap_{i\in I} J_i$ is an ideal. First note that $\bigcap_{i\in I} J_i$ is non-empty because $0 \in J_i$ for each $i \in I$, so $0 \in \bigcap_{i\in I} J_i$. Let $a, b \in \bigcap_{i\in I} J_i$. Then $a, b \in J_i$ for each $i \in I$, hence $a \vee b \in J_i$ for each $i \in I$ and so $a \vee b \in \bigcap_{i\in I} J_i$. Now let $x \in L$ and $y \in \bigcap_{i\in I} J_i$ such that $x \leqslant y$. Since $y \in J_i$ for each $i \in I$, it follows that $x \in J_i$ for each $i \in I$ and hence $x \in \bigcap_{i\in I} J_i$. Therefore $\bigcap_{i\in I} J_i \in \mathfrak{J}(L)$. $\square$

It follows that when L is a lattice with bottom 0, then $\mathfrak{J}(L)$ is a complete lattice in which, for any non-empty family $\{J_i\}_{i\in I} \subseteq \mathfrak{J}(L)$,

$$\bigwedge_{i\in I} J_i = \bigcap_{i\in I} J_i \ \text{ and } \ \bigvee_{i\in I} J_i = \bigcap\{K \in \mathfrak{J}(L) : \bigcup_{i\in I} J_i \subseteq K\}.$$

We will refer to $\mathfrak{J}(L)$ as the *ideal lattice* of L.

8.3 The Ideals of a Frame

Let L be a frame. Since L is a complete lattice, L has a bottom 0 and thus $\mathfrak{J}(L)$ is a complete lattice.

Proposition 8.3.1. *Let L be a frame and let $\{J_i\}_{i\in I} \subseteq \mathfrak{J}(L)$. Put*

$$K \coloneqq \{\bigvee F : F \subseteq L, F \text{ finite, and } F \subseteq \bigcup_{i\in I} J_i\}. \tag{8.1}$$

Then K is an ideal in L. (See [7], p. 131).

Proof. For each $i \in I$ we have $0 \in J_i$, hence $\{0\} \subseteq \bigcup_{i\in I} J_i$ and $\bigvee\{0\} = 0 \in K$. So K is indeed non-empty. Let $\bigvee F, \bigvee G \in K$. Then $(\bigvee F) \vee (\bigvee G) = \bigvee(F \cup G)$. Since F and G are both finite subsets of $\bigcup_{i\in I} J_i$, it follows that $F \cup G$ is a finite subset of $\bigcup_{i\in I} J_i$. Hence $\bigvee(F \cup G) \in K$ and it follows that K is closed under binary join. Now let $x \in L$ and $\bigvee F \in K$ be such that $x \leqslant \bigvee F$. Then $x = x \wedge \bigvee F = \bigvee\{x \wedge y : y \in F\}$. Now $\{x \wedge y : y \in F\}$ is a finite subset of L and, for $y \in F$, there exists $i \in I$ with $y \in J_i$ and (since $x \wedge y \leqslant y$) hence $x \wedge y \in J_i$. That is to say, for each $x \wedge y \in \{x \wedge y : y \in F\}$, there exists $i \in I$ with $x \wedge y \in J_i$. Hence $\{x \wedge y : y \in F\} \subseteq \bigcup_{i\in I} J_i$. Thus $x \in K$. We may conclude that K is an ideal in L. □

Proposition 8.3.2. *Let L be a frame, $\{J_i\}_{i\in I} \subseteq \mathfrak{J}(L)$ and K be as defined in Eq.*(8.1)*. Then $K = \bigvee_{i\in I} J_i$.*

Proof. We first show that K is an upper bound of $\{J_i\}_{i\in I}$. Take an $i \in I$ and let $x \in J_i$. Put $F = \{x\}$. Then F is finite and $F \subseteq \bigcup_{i\in I} J_i$ with $\bigvee F = x$. Hence $x \in K$. Therefore $J_i \subseteq K$. It follows that K is an upper bound of $\{J_i\}_{i\in I}$. Next we show that K is the least upper bound of $\{J_i\}_{i\in I}$. Let H be an ideal in L such that $J_i \subseteq H$ for all $i \in I$. Then $\bigcup_{i\in I} J_i \subseteq H$. Let $\bigvee G \in K$. Then $G \subseteq L$ is finite with $G \subseteq \bigcup_{i\in I} J_i$ and hence $G \subseteq H$. Now H is an ideal and is therefore closed under finite join, so $\bigvee G \in H$. Thus $K \subseteq H$. It follows that K is the least upper bound of $\{J_i\}_{i\in I}$. □

From the above it follows that, given any frame L, $\mathfrak{J}(L)$ is a complete lattice in which

$$\bigwedge_{i\in I} J_i = \bigcap_{i\in I} J_i \quad \text{and}$$
$$\bigvee_{i\in I} J_i = \{\bigvee F : F \subseteq L,\ F \text{ finite, and } F \subseteq \bigcup_{i\in I} J_i\}$$

for any non-empty collection $\{J_i\}_{i\in I} \subseteq \mathfrak{J}(L)$.

Proposition 8.3.3. *Let L be a frame. Then $\mathfrak{J}(L)$ is a frame.*

Proof. We need to verify that $\mathfrak{J}(L)$ satisfies the join-infinite distributivity law. Let $\{J_i\}_{i\in I} \subseteq \mathfrak{J}(L)$ and $A \in \mathfrak{J}(L)$. Then, by Lemma 3.2.2, we have

$$\bigvee_{i\in I} A \cap J_i \subseteq A \cap \bigvee_{i\in I} J_i.$$

Let $x \in A \cap \bigvee_{i\in I} J_i$. Then $x = \bigvee F$ in L for some finite $F \subseteq \bigcup_{i\in I} J_i$. Now $\bigvee F \in A$ and, for each $y \in F$, $y \leqslant \bigvee F$ and hence $y \in A$ – so $F \subseteq A$. Thus $F \subseteq A \cap \bigcup_{i\in I} J_i = \bigcup_{i\in I} A \cap J_i$. It now follows that

$$x = \bigvee F \in \{\bigvee G : G \subseteq L,\ G \text{ finite}, G \subseteq \bigcup_{i\in I} A \cap J_i\} = \bigvee_{i\in I} A \cap J_i.$$

Hence $A \cap \bigvee_{i\in I} J_i \subseteq \bigvee_{i\in I} A \cap J_i$, and it follows that $A \cap \bigvee_{i\in I} J_i = \bigvee_{i\in I} A \cap J_i$. Therefore $\mathfrak{J}(L)$ satisfies the join-infinite distributivity law and we can conclude that that $\mathfrak{J}(L)$ is a frame. □

8.4 The Axiom of Choice and the Ascending Chain Condition

Now that we have shown that the ideal lattice of a frame is a frame, we can proceed to find a sufficient condition for the ideal lattice to be spatial. In what follows we will need the Axiom of Choice ([2], p. 229).

Axiom of Choice. For any non-empty collection $\{A_i\}_{i\in I}$ of non-empty sets, there exists a function $f : I \longrightarrow \bigcup_{i\in I} A_i$ with the property that $f(i) \in A_i$ for each $i \in I$.

We want to identify the conditions under which the ideal lattice of a frame is spatial. Thus for any frame L we must find the properties of L which make $\mathcal{J}(L)$ spatial. We now introduce a definition.

Definition 8.4.1 (Ascending Chain Condition)**.** Let P be a poset. We say that P satisfies the **ascending chain condition** (ACC) iff, given any monotonically increasing sequence $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$ of elements in P, there exists $k \in \mathbb{N}$ such that $x_m = x_k$ for all natural numbers $m \geqslant k$ ([2], p. 51).

We now present a lemma which is dependent upon the Axiom of Choice, for its proof we will prove the contrapositive in both directions.

Lemma 8.4.2. *Let P be a non-empty poset. Then P satisfies the ACC iff every non-empty subset of P has a maximal element.* (See [2], p. 52).

Proof. ($\Rightarrow$) Suppose there exists a non-empty subset A of P such that A does not have a maximal element. Then, for each $x \in A$, there exists $y \in A$ such that $x < y$. For each $x \in A$, put $A_x \coloneqq \{y \in A : x < y\}$. Then $\{A_x\}_{x\in A}$ is a non-empty family of non-empty subsets of A. By the Axiom of Choice, there exists a function $f : A \longrightarrow \bigcup_{x\in A} A_x$ such that $f(x) \in A_x$ for each $x \in A$. Now fix $x_1 \in A$ and, for each $n \in \mathbb{N}$, define $x_{n+1} = f(x_n)$. This defines a strictly increasing chain in A: $x_1 < x_2 < x_3 < \ldots$. Therefore P does not satisfy the ACC.

($\Leftarrow$) Suppose that P does not satisfy the ACC. Then there exists an infinite ascending chain in P: $x_1 < x_2 < x_3 < \ldots$. Put $A \coloneqq \{x_n : n \in \mathbb{N}\}$. Then A does not have a maximal element. □

Lemma 8.4.3. *Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$ be a chain of ideals in a lattice L. Then $\bigcup_{n\in\mathbb{N}} J_n$ is an ideal in L.*

Proof. Each J_n is non-empty, hence $\bigcup_{n\in\mathbb{N}} J_n$ is non-empty. Let $x, y \in \bigcup_{n\in\mathbb{N}} J_n$. Then $x \in J_k$ and $y \in J_\ell$ for some $k, \ell \in \mathbb{N}$. Without loss of generality, suppose $k \leqslant \ell$. Then $J_k \subseteq J_\ell$ and hence $x \in J_\ell$. Thus $x \vee y \in J_\ell$, which implies that $x \vee y \in \bigcup_{n\in\mathbb{N}} J_n$. Let $a \in L$ and $b \in \bigcup_{n\in\mathbb{N}} J_n$ such that $a \leqslant b$. Then $b \in J_p$ for some $p \in \mathbb{N}$ and hence $a \in J_p$. Thus $a \in \bigcup_{n\in\mathbb{N}} J_n$. We may conclude that $\bigcup_{n\in\mathbb{N}} J_n$ is an ideal in L. □

Proposition 8.4.4. *Let L be a lattice. Then L satisfies the ACC iff every ideal in L is principal.*

Proof. ($\Rightarrow$) Suppose that L satisfies the ACC. Let J be an ideal in L. Then J has a maximal element, say $a \in J$. We claim that this $a \in J$ is the only maximal element in J. Let $b \in J$ with $b \neq a$ and suppose that b is maximal in J. Then $a \not< b$ (since a is maximal) and $b \not< a$ (since b is maximal). Thus $a \parallel b$ (the symbol $\parallel$ denotes incomparability). Since J is an ideal, we have $a \vee b \in J$. Now since $a \parallel b$, we have $a < a \vee b$ and $b < a \vee b$. But this contradicts the maximality of a and b in J. Therefore it must be that b is not maximal in J and it follows that a is the only maximal element in J. Hence, for all $x \in J$, $a \vee x = a$ (as we do not have

$a \vee x > a$ by maximality of a in J) and it follows that $x \leqslant a$. Therefore if $x \in J$ then $x \leqslant a$, and (conversely) if $x \in L$ is such that $x \leqslant a$ then $x \in J$ (because J is a down-set). We can conclude that, for all $x \in L$, $x \in J$ iff $x \leqslant a$. Hence $J = \downarrow a$.

($\Leftarrow$) We will prove the contrapositive. Suppose that L does not satisfy the ACC. Then there exists an infinite ascending chain $x_1 < x_2 < \ldots < x_n < \ldots$ in L. Then we can construct an infinite ascending chain of ideals $\downarrow x_1 \subsetneq \downarrow x_2 \subsetneq \ldots \subsetneq \downarrow x_n \subsetneq \ldots$. By Lemma 8.4.3 it follows that $J := \bigcup_{n \in \mathbb{N}} \downarrow x_n$ is an ideal in L. Explicitly, $J = \{x \in L : (\exists n \in \mathbb{N})\ x \leqslant x_n\}$. Suppose that J is principal. Then there exists $a \in J$ such that $x \leqslant a$ for all $x \in J$. Since $a \in J$, there exists $n \in \mathbb{N}$ such that $a \leqslant x_n$. But $x_{n+1} \in J$ and $x_n < x_{n+1}$, so $a < x_{n+1}$ – a contradiction. Therefore it must be that J is not principal. $\square$

Proposition 8.4.5. *Let L be a lattice with bottom* 0. *Then L satisfies the ACC iff $\mathcal{J}(L)$ satisfies the ACC.*

Proof. ($\Rightarrow$) Suppose that L satisfies the ACC. Then every ideal in L is principal. Let $J_1 \subseteq J_2 \subseteq \ldots \subseteq J_n \subseteq \ldots$ be an increasing chain in $\mathcal{J}(L)$. Then there exists $\{x_\ell\}_{\ell \in \mathbb{N}} \subseteq L$ such that $J_\ell = \downarrow x_\ell$ for each $\ell \in \mathbb{N}$. Then $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$ is an ascending chain in L and thus there exists $k \in \mathbb{N}$ such that $x_m = x_k$ for all natural numbers $m \geqslant k$. Therefore $\downarrow x_m = \downarrow x_k$ for all natural numbers $m \geqslant k$, hence $J_m = J_k$ for all natural numbers $m \geqslant k$. We may conclude that $\mathcal{J}(L)$ satisfies the ACC.

($\Leftarrow$) Suppose that $\mathcal{J}(L)$ satisfies the ACC. Let $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$ be an increasing chain in L. Then $\downarrow x_1 \subseteq \downarrow x_2 \subseteq \ldots \subseteq \downarrow x_n \subseteq \ldots$ is an increasing chain in $\mathcal{J}(L)$. Now there exists $k \in \mathbb{N}$ such that $\downarrow x_m = \downarrow x_k$ for all natural numbers $m \geqslant k$, hence $x_m = x_k$ for all natural numbers $m \geqslant k$. Therefore L satisfies the ACC. $\square$

8.5 A Sufficient Condition for $\mathcal{J}(L)$ to be Spatial

As we did for frames, for a lattice L we will denote the set of meet-irreducible elements in L by $\mathcal{M}(L)$.

Lemma 8.5.1. *Let L be a lattice which satisfies the ACC. Let $a, b \in L$ be such that $b \nleqslant a$. Then there exists $p \in \mathcal{M}(L)$ such that $a \leqslant p$ and $b \nleqslant p$.* (cf. [2], p. 55).

Proof. Put $A := \{x \in L : a \leqslant x \text{ and } b \nleqslant x\}$. Note that A is non-empty because $a \in A$. By Lemma 8.4.2, A has a maximal element $p \in A$. We claim that this p is meet-irreducible. If L has a top $1 \in L$, then clearly $p \neq 1$ because $b \nleqslant p$. Suppose that p is not meet-irreducible. Then there exist $c, d \in L$ such that $p = c \wedge d$ and $c, d \neq p$. So $p < c$ and $p < d$. By the maximality of p in A, we must have $c, d \notin A$. Since $a \leqslant p$, we have $a < c$ and $a < d$, but $c, d \notin A$ so it must be that $b \leqslant c$ and $b \leqslant d$. Therefore $b \leqslant c \wedge d = p$, a contradiction. It must be that p is meet-irreducible. $\square$

Proposition 8.5.2. *Let L be a lattice which satisfies the ACC. Then, for each $a \in L$, we have $a = \bigwedge\{p \in \mathcal{M}(L) : a \leqslant p\}$.* (cf. [2], p. 55).

Proof. Let $a \in L$ and put $A := \{p \in \mathcal{M}(L) : a \leqslant p\}$. Clearly a is a lower bound of A. Let $b \in L$ be a lower bound of A. Suppose that $b \nleqslant a$. Then $a \vee b \neq a$, so we have $a < a \vee b$ and hence $a \vee b \nleqslant a$. By Lemma 8.5.1, there exists $p \in \mathcal{M}(L)$ such that $a \leqslant p$ and $a \vee b \nleqslant p$. Hence $p \in A$ and it follows that $b \leqslant p$. But then $a \vee b \leqslant p$, a contradiction. It must therefore be the case that $b \leqslant a$, and hence a is the greatest lower bound of A. $\square$

Bearing in mind our characterisation for spatial frames (Theorem 7.2.2), Proposition 8.5.2 implies that any frame L which satisfies the ACC is spatial. By Proposition 8.4.5, a frame L satisfies the ACC iff $\mathfrak{J}(L)$ satisfies the ACC, and thus if a frame L satisfies the ACC then $\mathfrak{J}(L)$ is spatial. We summarise these results as corollaries.

Corollary 8.5.3. *Every frame L which satisfies the ACC is spatial.*

Corollary 8.5.4. *If L is a frame which satisfies the ACC, then $\mathfrak{J}(L)$ is spatial.*

This provides our sufficient condition for the ideal lattice of a frame to be spatial. We note that this result explicitly needed the Axiom of Choice; we first used the Axiom of Choice in the proof of Lemma 8.4.2 – which was subsequently used in the proof of Lemma 8.5.1.

9 Conclusions

We have motivated the study of point-free topology from the perspective that frames are considered to be generalised spaces. We hope that this report has achieved in conveying some of the basic elements and results in point-free topology, and that the process of moving from classical topology to the point-free setting is a natural one. Indeed the spectrum adjunction facilitates the transition between classical topology and point-free topology.

Our goal for future work is to further study the spatiality of the ideal lattice in the context of coherent frames (a frame is called coherent iff it is isomorphic to the ideal lattice of some distributive lattice) ([3], p. 64). Johnstone ([3], p. 65) presents a theorem which states that every coherent frame is spatial – according to Banaschewski [1] this is in fact equivalent to the Boolean Ultrafilter Theorem. Thus one may show that the ideal lattice of *any* frame is spatial (assuming the Boolean Ultrafilter Theorem).

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